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ANALYTICAL PHENOMENON

CARTEA ROMÂNEASCĂ
EDUCAŢIONAL

MOTTO:

"The best way to follow your dreams
is to wake up!"

Table of Contents

Chapter O – Famous Theorems 6

Chapter I – Questions 27

Chapter II – Solutions 62

Bibliography 237

Chapter 0
FAMOUS THEOREMS

CAUCHY–SCHWARZ Inequality

$$(ax+by)^2 \le (a^2+b^2)(x^2+y^2);\ a,b,x,y \in \mathbb{R}$$

$$(ax+by+cz)^2 \le (a^2+b^2+c^2)(x^2+y^2+z^2);\ a,b,c,x,y,z \in \mathbb{R}$$

$$\left(\sum_{i=1}^{n} a_i x_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} x_i^2\right); a_i, x_i \in \mathbb{R}, i \in \overline{1,n}$$

$$\frac{a^2}{x}+\frac{b^2}{y} \ge \frac{(a+b)^2}{x+y}; a,b \in \mathbb{R}; x,y \in (0,\infty)$$

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z} \ge \frac{(a+b+c)^2}{x+y+z}; a,b,c \in \mathbb{R}; x,y,z \in (0,\infty)$$

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z} \ge \frac{(a+b+c)^2}{ax+by+cz}; a,b,c,x,y,z \in (0,\infty)$$

$$\frac{a_1^2}{x_1}+\frac{a_2^2}{x_2}+\cdots+\frac{a_n^2}{x_n} \ge \frac{(a_1+a_2+\cdots+a_n)^2}{x_1+x_2+\cdots+x_n}; a_i \in \mathbb{R}; x_i > 0; i \in \overline{1,n}$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}; (\forall) a,b,c \in (0,\infty)$$

$$\frac{a}{b+nc}+\frac{b}{c+na}+\frac{c}{a+nb} \ge \frac{3}{n+1}; a,b,c \in (0,\infty);\ n \in \mathbb{N}^*$$

MINKOWSKI's Inequality

$$\sqrt{(x+a)^2+(y+b)^2} \le \sqrt{x^2+y^2}+\sqrt{a^2+b^2}$$

$$\sqrt{(x+y+z)^2+(a+b+c)^2} \le \sqrt{x^2+a^2}+\sqrt{y^2+b^2}+\sqrt{z^2+c^2}$$

$$\sqrt{(x+a)^2+(y+b)^2+(z+c)^2} \le \sqrt{x^2+y^2+z^2}+\sqrt{a^2+b^2+c^2}$$

$$\sqrt{(x_1+a_1)^2+(x_2+a_2)^2+\cdots+(x_n+a_n)^2} \le \sqrt{x_1^2+x_2^2+\cdots+x_n^2}+\sqrt{a_1^2+a_2^2+\cdots+a_n^2}$$

$$x_i; a_i \in \mathbb{R}; i \in \overline{1,n}; n \in \mathbb{N}^*$$

$$\left(\sum_{i=1}^{n} |x_i+y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

$$p > 1; x_i, y_i \in \mathbb{R}; i \in \overline{1,n}; n \in \mathbb{N}^*$$

HÖLDER's Inequality

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z} \ge \frac{(a+b+c)^3}{3(x+y+z)}; a,b,c,x,y,z \in (0,\infty)$$

$$\frac{a^4}{x}+\frac{b^4}{y}+\frac{c^4}{z}\geq\frac{(a+b+c)^4}{9(x+y+z)}; a,b,c,x,y,z\in(0,\infty)$$

$$\frac{a^n}{x}+\frac{b^n}{y}+\frac{c^n}{z}\geq\frac{(a+b+c)^n}{3^{n-2}(x+y+z)}; a,b,c,x,y,z\in(0,\infty); n\in\mathbb{N}; n\geq 2$$

$$\frac{a^n}{x}+\frac{b^n}{y}\geq\frac{(a+b)^n}{2^{n-2}(x+y)}; a,b,x,y\in(0,\infty);\ n\geq 2; n\in\mathbb{N}$$

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}\geq\frac{(x+y+z)^3}{(a+b+c)^2}; x,y,z,a,b,c\in(0,\infty)$$

$$\frac{x^4}{a^3}+\frac{y^4}{b^3}+\frac{z^4}{c^3}\geq\frac{(x+y+z)^4}{(a+b+c)^3}; x,y,z,a,b,c\in(0,\infty)$$

$$\frac{x^{n+1}}{a^n}+\frac{y^{n+1}}{b^n}+\frac{z^{n+1}}{c^n}\geq\frac{(x+y+z)^{n+1}}{(a+b+c)^n}; x,y,z,a,b,c\in(0,\infty);\ n\in\mathbb{N}$$

$$\left(\sum_{i=1}^{n}a_i^3\right)\left(\sum_{i=1}^{n}b_i^3\right)\left(\sum_{i=1}^{n}c_i^3\right)\geq\left(\sum_{i=1}^{n}a_ib_ic_i\right)^3; a_i,b_i,c_i\in[0,\infty); n\in\mathbb{N}^*$$

$$\left(\sum_{i=1}^{n}a_i^4\right)\left(\sum_{i=1}^{n}b_i^4\right)\left(\sum_{i=1}^{n}c_i^4\right)\left(\sum_{i=1}^{n}d_i^4\right)\geq\left(\sum_{i=1}^{n}a_ib_ic_id_i\right)^4; a_i,b_i,c_i,d_i\in\mathbb{R}; n\in\mathbb{N}^*$$

$$\sum_{i=1}^{n}|x_iy_i|\leq\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}\cdot\left(\sum_{i=1}^{n}|y_i|^q\right)^{\frac{1}{q}}; p,q\in(1,\infty)$$

$$\frac{1}{p}+\frac{1}{q}=1; x_i,y_i\in\mathbb{R}; i\in\overline{1,n}; n\in\mathbb{N}^*$$

HUYGENS's Inequality

$$(1+a_1)(1+a_2)\geq\left(1+\sqrt{a_1a_2}\right)^2; a_1,a_2\in[0,\infty)$$

$$(1+a_1)(1+a_2)(1+a_3)\geq\left(1+\sqrt[3]{a_1a_2a_3}\right)^3; a_1,a_2,a_3\in[0,\infty)$$

$$(1+a_1)(1+a_2)(1+a_3)(1+a_4)\geq\left(1+\sqrt[4]{a_1a_2a_3a_4}\right)^4; a_1,a_2,a_3,a_4\in[0,\infty)$$

$$\prod_{i=1}^{n}(1+x_i)\geq\left(1+\sqrt[n]{x_1x_2\cdots x_n}\right)^n; a_i\in[0,\infty);\ n\in\mathbb{N}; n\geq 2$$

$$(a_1+b_1)(a_2+b_2)\geq\left(\sqrt{a_1a_2}+\sqrt{b_1b_2}\right)^2$$

$$(a_1+b_1)(a_2+b_2)(a_3+b_3)\geq\left(\sqrt[3]{a_1a_2a_3}+\sqrt[3]{b_1b_2b_3}\right)^3$$

$$(a_1+b_1)(a_2+b_2)\cdot\ldots\cdot(a_n+b_n)\geq\left(\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}+\sqrt[n]{b_1b_2\cdot\ldots\cdot b_n}\right)^n$$

$$(a_1+b_1+c_1)(a_2+b_2+c_2)\geq\left(\sqrt{a_1a_2}+\sqrt{b_1b_2}+\sqrt{c_1c_2}\right)^2$$

$$(a_1+b_1+c_1)(a_2+b_2+c_2)\cdot\ldots\cdot(a_n+b_n+c_n)$$
$$\geq\left(\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}+\sqrt[n]{b_1b_2\cdot\ldots\cdot b_n}+\sqrt[n]{c_1c_2\cdot\ldots\cdot c_n}\right)^n$$

Generalization of HÖLDER's Inequality

$$\prod_{i=1}^{n}\left(\sum_{j=1}^{n} x_{ij}\right)^{w_j} \geq \sum_{i=1}^{n}\left(\prod_{j=1}^{n} x_{ij}^{w_i}\right)$$

$$w_1 + w_2 + \cdots + w_n = 1$$

CHEBYSHEV's Inequality

$$\begin{cases} (x_1 \leq x_2) \wedge (y_1 \leq y_2) \text{ or } (x_1 \geq x_2) \wedge (y_1 \geq y_2) \\ x_1y_1 + x_2y_2 \geq \frac{1}{2}(x_1 + x_2)(y_1 + y_2) \end{cases}$$

$$\begin{cases} (x_1 \leq x_2 \leq x_3) \wedge (y_1 \leq y_2 \leq y_3) \text{ or } (x_1 \geq x_2 \geq x_3) \wedge (y_1 \geq y_2 \geq y_3) \\ x_1y_1 + x_2y_2 + x_3y_3 \geq \frac{1}{3}(x_1 + x_2 + x_3)(y_1 + y_2 + y_3) \end{cases}$$

$$\begin{cases} (x_1 \leq x_2 \leq \cdots \leq x_n) \wedge (y_1 \leq y_2 \leq \cdots \leq y_n) \text{ or } (x_1 \geq x_2 \geq \cdots \geq x_n) \wedge (y_1 \geq y_2 \geq \cdots \geq y_n) \\ x_1y_1 + x_2y_2 + \cdots + x_ny_n \geq \frac{1}{n}(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) \end{cases}$$

$$\begin{cases} (x_1 \leq x_2) \wedge (y_1 \geq y_2) \text{ or } (x_1 \geq x_2) \wedge (y_2 \leq y_2) \\ x_1y_1 + x_2y_2 \leq \frac{1}{2}(x_1 + x_2)(y_1 + y_2) \end{cases}$$

$$\begin{cases} (x_1 \leq x_2 \leq x_3) \wedge (y_1 \geq y_2 \geq y_3) \text{ or } (x_1 \geq x_2 \geq x_3) \wedge (y_1 + y_2 + y_3) \\ x_1y_1 + x_2y_2 + x_3y_3 \leq \frac{1}{3}(x_1 + x_2 + x_3)(y_1 + y_2 + y_3) \end{cases}$$

$$\begin{cases} (x_1 \leq x_2 \leq \cdots \leq x_n) \wedge (y_1 \geq y_2 \geq \cdots \geq y_n) \text{ or } (x_1 \geq x_2 \geq \cdots \geq x_n) \wedge \\ \wedge (y_1 \leq y_2 \leq \cdots \leq y_n) \\ x_1y_1 + x_2y_2 + \cdots + x_ny_n \leq \frac{1}{n}(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) \\ x_i, y_i \in \mathbb{R}; n \in \mathbb{N}^*; i \in \overline{1,n} \end{cases}$$

$$\begin{cases} \sum_{i=1}^{n} f(a_i)g(b_i)p_i \geq \left(\sum_{i=1}^{n} f(a_i)p_i\right)\left(\sum_{i=1}^{n} g(b_i)p_i\right) \geq \sum_{i=1}^{n} f(a_i)g(b_{n-i+1})\,p_i \\ \text{for } a_1 \leq a_2 \leq \cdots \leq a_n; b_1 \leq b_2 \leq \cdots \leq b_n \\ p_i \geq 0; i \in \overline{1,n}; n \in \mathbb{N}^*; p_1 + p_2 + \cdots + p_n = 1 \end{cases}$$

f, g non-decreasing

SCHUR's Inequality

$$\begin{cases} a^r(a-b)(a-c)+b^r(b-a)(b-c)+c^r(c-a)(c-b)\ge 0 \\ a,b,c\in[0,\infty); r>0 \end{cases}$$

$$\begin{cases} a^3+b^3+c^3+3abc\ge ab(a+b)+bc(b+c)+ca(c+a) \\ abc\ge(-a+b+c)(a-b+c)(a+b-c) \\ (a+b+c)^3+9abc\ge 4(a+b+c)(ab+bc+ca) \\ (a-b)^2(a+b-c)+(b-c)^2(b+c-a)+(c-a)^2(c+a-b)\ge 0 \\ a^2+b^2+c^2+\dfrac{9abc}{a+b+c}\ge 2(ab+bc+ca) \\ \dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}+\dfrac{4abc}{(a+b)(b+c)(c+a)}\ge 2 \end{cases}$$
$$a,b,c\in(0,\infty)$$

$$\begin{cases} a^4+b^4+c^4+abc(a+b+c)\ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) \\ a,b,c\in[0,\infty) \end{cases}$$

$$\begin{cases} \displaystyle\sum_{i=1}^{n}\lambda_i^2\le\sum_{i=1}^{n}\left(\sum_{j=1}^{n}A_{ij}^2\right) \qquad ; A\in M_n(\mathbb{R}) \\ \lambda_1,\lambda_2,\cdots,\lambda_n \text{ eigenvalues of } A; n\in\mathbb{N}^* \end{cases}$$

$$\begin{cases} \displaystyle\sum_{i=1}^{k}a_{ii}\le\sum_{i=1}^{k}\lambda_i\,; 1\le k\le n; n\in\mathbb{N}^* \\ \lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n \text{ eigenvalues of } A=\left(a_{ij}\right) \\ a_{11}\ge a_{22}\ge\cdots\ge a_{nn} \end{cases}$$

MILNE's Inequality

$$(a_1+b_1+a_2+b_2)\left(\frac{a_1b_1}{a_1+b_1}+\frac{a_2b_2}{a_2+b_2}\right)\le(a_1+a_2)(b_2+b_2)$$

$$(a_1+b_1+a_2+b_2+a_3+b_3)\left(\frac{a_1b_1}{a_1+b_1}+\frac{a_2b_2}{a_2+b_2}+\frac{a_3b_3}{a_3+b_3}\right)$$
$$\le(a_1+a_2+a_3)\cdot(b_1+b_2+b_3)$$

$$\left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right)\le\left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}b_i\right), i\in\overline{1,n}; n\ge 2;$$
$$a_i>0; b_i>0$$

REARRANGEMENT's Inequality

$$\begin{cases} \sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} a_i b_{n-i+1} \\ a_1 \leq a_2 \leq \cdots \leq a_n; b_1 \leq b_2 \leq \cdots \leq b_n \\ \pi = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ \ldots & \ldots & \pi(i) & \ldots & \ldots & \ldots \end{pmatrix} \in S_n; n \in \mathbb{N}^* \end{cases}$$

$$\begin{cases} \sum_{i=1}^{n} f_i(b_i) \geq \sum_{i=1}^{n} f_i\left(b_{\pi(i)}\right) \geq \sum_{i=1}^{n} f_i(b_{n-i+1}) \\ \text{with } \left(f_{i+1}(x) - f_i(x)\right) \text{ non-decreasing; } 1 \leq i \leq n \end{cases}$$

STIRLING's Inequality

$$e\left(\frac{n}{e}\right)^n \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \leq en\left(\frac{n}{e}\right)^n$$

MAHLER's Inequality

$$\sqrt{(x_1 + y_1)(x_2 + y_2)} \geq \sqrt{x_1 x_2} + \sqrt{y_1 y_2}$$

$$\sqrt[3]{(x_1 + y_1)(x_2 + y_2)(x_3 + y_3)} \geq \sqrt[3]{x_1 x_2 x_3} + \sqrt[3]{y_1 y_2 y_3}$$

$$\sqrt[4]{(x_1 + y_1)(x_2 + y_2)(x_3 + y_3)(x_4 + y_4)} \geq \sqrt[4]{x_1 x_2 x_3 x_4} + \sqrt[4]{y_1 y_2 y_3 y_4}$$

$$\prod_{i=1}^{n} (x_i + y_i)^{\frac{1}{n}} \geq \prod_{i=1}^{n} x_i^{\frac{1}{n}} + \prod_{i=1}^{n} y_i^{\frac{1}{n}}$$

$$x_i, y_i > 0; i \in \overline{2, n}; n \in \mathbb{N}; n \geq 2$$

WEIERSTRASS Inequality

$$\prod_{i=1}^{n} (1 - x_i)^{w_i} \geq 1 - \sum_{i=1}^{n} w_i x_i \,; x_i \leq 1; w_i \geq 1 \text{ or } w_i \leq 0; i \in \overline{1, n}$$

$$\prod_{i=1}^{n} (1 - x_i)^{w_i} \leq 1 - \sum_{i=1}^{n} w_i x_i \,; w_i \in [0,1]; \sum_{i=1}^{n} w_i \leq 1\,;$$

$$x_i \in (-\infty, 1]; \; i \in \overline{1, n}; n \in \mathbb{N}^*$$

RADON's Inequality

$$\frac{x_1^m}{a_1^{m-1}} + \frac{x_2^m}{a_2^{m-1}} + \cdots + \frac{x_n^m}{a_n^{m-1}} \geq \frac{(x_1 + x_2 + \cdots + x_n)^m}{(a_1 + a_2 + \cdots + a_n)^{m-1}}$$

$$x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_n \in (0, \infty); m \geq 1; n \geq 1$$

YOUNG's Inequality

$$\left(\frac{1}{px^p}+\frac{1}{qx^q}\right)^{-1} \le xy \le \frac{x^p}{p}+\frac{y^q}{q}, x,y \in (0,\infty);\ p,q \in (0,\infty); \frac{1}{p}+\frac{1}{q}=1$$

JENSEN's Inequality

$$\begin{cases} f: I \to \mathbb{R}; f \text{ convex on } I; \\ a,b,c,x_1x_2,\cdots,x_n \in I; n \in \mathbb{N}^* \\ f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} \\ f\left(\frac{a+b+c}{3}\right) \le \frac{f(a)+f(b)+f(c)}{3} \\ f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \le \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \end{cases}$$

$$\begin{cases} f: I \to \mathbb{R}; f \text{ concave on } I; \\ f\left(\frac{a+b}{2}\right) \ge \frac{f(a)+f(b)}{2} \\ f\left(\frac{a+b+c}{3}\right) \ge \frac{f(a)+f(b)+f(c)}{3} \\ f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \ge \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \end{cases}$$

$$\begin{cases} f: I \to \mathbb{R}; f \text{ convex on } I; \\ p_i \ge 0; p_1+p_2+\cdots+p_n=1 \\ f(p_1x_1+p_2x_2+\cdots+p_nx_n) \le p_1f(x_1)+p_2f(x_2)+\cdots+p_nf(x_n) \end{cases}$$

$$\begin{cases} f: I \to \mathbb{R}; f \text{ concave on } I \\ p_i \ge 0; p_1+p_2+\cdots+p_n=1 \\ f(p_1x_1+p_2x_2+\cdots+p_nx_n) \ge p_1f(x_1)+p_2f(x_2)+\cdots+p_nf(x_n) \end{cases}$$

WEIGHTED MEANS Inequality

$$\begin{cases} w = w_1+w_2 \\ \frac{w_1x_1+w_2x_2}{w_1+w_2} \ge \sqrt[w_1+w_2]{x_1^{w_1}\cdot x_2^{w_2}} \ge \frac{w}{\frac{w_1}{x_1}+\frac{w_2}{x_2}} \end{cases}$$

$$\begin{cases} w = w_1+w_2+w_3 \\ \frac{w_1x_1+w_2x_2+w_3x_3}{w_1+w_2+w_3} \ge \sqrt[w_1+w_2+w_3]{x_1^{w_1}\cdot x_2^{w_2}\cdot x_3^{w_3}} \ge \frac{w}{\frac{w_1}{x_1}+\frac{w_2}{x_2}+\frac{w_3}{x_3}} \end{cases}$$

$$\begin{cases} \dfrac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} \geq \sqrt[\sum_{i=1}^{n} w_i]{\prod_{i=1}^{n} x_i^{w_i}} \geq \dfrac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} \frac{w_i}{x_i}} \\ w_i > 0; i \in \overline{1,n}; n \in \mathbb{N}^*. \text{Convention: } 0^0 = 1 \end{cases}$$

MEANS Inequality

$$\begin{cases} \min\{x_i\} \leq \dfrac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \leq \sqrt[n]{\prod_{i=1}^{n} x_i} \leq \dfrac{\sum_{i=1}^{n} x_i}{n} \leq \sqrt{\dfrac{1}{n}\sum_{i=1}^{n} x_i^2} \leq \max\{x_i\} \\ x_i \in (0,\infty); i \in \overline{1,n}; i \in \overline{1,n}; n \in \mathbb{N}^* \end{cases}$$

POWER MEANS Inequality

$$\begin{cases} \sqrt{a|x_1|^2 + b|x_2|^2} \leq \sqrt[3]{a|x_1|^3 + b|x_2|^3} \\ a, b, \in [0,\infty); a + b = 1; x_1, x_2 \in \mathbb{R} \end{cases}$$

$$\begin{cases} \sqrt[p]{a|x_1|^p + b|x_2|^p} \leq \sqrt[q]{a|x_1|^9 + b|x_2|^q} \\ p \geq q > 0; a, b \in [0,\infty); a + b = 1; x_1, x_2 \in \mathbb{R} \end{cases}$$

$$\begin{cases} \sqrt{a|x_1|^2 + b|x_2|^2 + c|x_3|^2} \leq \sqrt[3]{a|x_1|^3 + b|x_2|^3 + c|x_3|^3} \\ a, b, c \in [0,\infty); a + b + c = 1; x_1, x_2, x_3 \in \mathbb{R} \end{cases}$$

$$\begin{cases} \sqrt[p]{a|x_1|^2 + b|x_2|^2 + c|x_3|^2} \leq \sqrt[q]{a|x_1|^3 + b|x_2|^3 + c|x_3|^3} \\ a, b, c \in [0,\infty); a + b + c = 1; x_1, x_2, x_3 \in \mathbb{R} \end{cases}$$

$$\begin{cases} \sqrt[p]{\sum_{i=1}^{n} w_i |x_i|^p} \leq \sqrt[q]{\sum_{i=1}^{n} w_i |x_i|^q} \\ p, q \in [0,\infty); p \leq q; w_i \in [0,\infty) \\ i \in \overline{1,n}; n \in \mathbb{N}^*; w_1 + w_2 + \cdots + w_n = 1 \\ M(p) = \sqrt[p]{\sum_{i=1}^{n} w_i |x_i|^p} \\ \lim_{p \to 0} M(p) = \prod_{i=1}^{n} |x_i|^{w_i} \\ \lim_{x \to -\infty} M(p) = \min\{x_1, x_2, \cdots, x_n\} \\ \lim_{x \to \infty} M(p) = \max\{x_1, x_2, \ldots, x_n\} \end{cases}$$

LEHMER's Inequality

$$\frac{w_1|x_1|^p + w_2|x_2|^p}{w_1|x_1|^{p-1} + w_2|x_2|^{p-1}} \le \frac{w_1|x_1|^q + w_2|x_2|^q}{w_1|x_1|^{q-1} + w_2|x_2|^{q-1}}$$

$$\frac{w_1|x_1|^p + w_2|x_2|^p + w_3|x_3|^p}{w_1|x_1|^{p-1} + w_2|x_2|^{p-1} + w_3|x_3|^{p-1}} \le \frac{w_1|x_1|^q + w_2|x_2|^q + w_3|x_3|^q}{w_1|x_1|^{q-1} + w_2|x_2|^{q-1} + w_3|x_3|^{q-1}}$$

$$\frac{\sum_{i=1}^n w_i|x_i|^p}{\sum_{i=1}^n w_i|x_i|^{p-1}} \le \frac{\sum_{i=1}^n w_i|x_i|^q}{\sum_{i=1}^n w_i|x_i|^{q-1}}, p \le q; w_i \ge 0; i \in \overline{1,n}; n \in \mathbb{N}$$

CARLEMAN's Inequality

$$\sum_{k=1}^n \left(\prod_{i=1}^k |a_i|\right)^{\frac{1}{k}} \le e\sum_{k=1}^n |a_k|, n \in \mathbb{N}^*;\ a_1, a_2, \dots, a_n \in \mathbb{R}$$

CALLEBAUT's Inequality

$$\left(\sum_{i=1}^n a_i^{1+x} b_i^{1-x}\right)\left(\sum_{i=1}^n a_i^{1-x} b_i^{1+x}\right) \ge \left(\sum_{i=1}^n a_i^{1+y} b_i^{1-y}\right)\left(\sum_{i=1}^n a_i^{1-y} b_i^{1+y}\right)$$

$$1 \ge x \ge y \ge 0; i \in \overline{1,n}; n \in \mathbb{N}^*$$

SUM & PRODUCT Inequality

$$\begin{cases} \sum_{j=1}^m \prod_{i=1}^n a_{ij} \ge \sum_{j=1}^m \prod_{i=1}^n a_{i\pi(j)} \\ \prod_{j=1}^m \sum_{i=1}^n a_{ij} \le \prod_{j=1}^m \sum_{i=1}^n a_{i\pi(j)} \\ 0 \le a_{i1} \le a_{i2} \le \dots \le a_{im}; i \in \overline{1,n} \\ \pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & i & \cdots & n \\ \dots & \dots & \dots & \pi(i) & \dots & \dots & \dots \end{pmatrix} \end{cases}$$

$$\begin{cases} \left|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i\right| \le \sum_{i=1}^n |a_i - b_i|; |a_i| \le 1; |b_i| \le 1 \\ i \in \overline{1,n}; n \in \mathbb{N}^*; a_i, b_i \in \mathbb{R}; \end{cases}$$

$$\prod_{i=1}^n (\alpha + a_i) \ge (1+\alpha)^n; \prod_{i=1}^n a_i \ge 1; a_i > 0; \alpha > 0$$

SQUARE & ROOT Inequality

$$\left\{2\sqrt{x+1}-2\sqrt{x}<\frac{1}{\sqrt{x}}<\sqrt{x+1}-\sqrt{x-1}<2\sqrt{x}-2\sqrt{x-1}; x\geq 1\right.$$

LOGARITHM MEAN Inequality

$$\begin{cases}\sqrt{xy}\leq\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)\sqrt[4]{xy}\leq\frac{x-y}{\ln x-\ln y}\leq\frac{\left(\sqrt{x}+\sqrt{y}\right)^2}{2}\leq\frac{x+y}{2};\\ x,y\in(0,\infty)\end{cases}$$

HEINZ's Inequality

$$\left\{\sqrt{xy}\leq\frac{x^{1-\alpha}y^{\alpha}+x^{\alpha}y^{1-\alpha}}{2}\leq\frac{x+y}{2}; x,y>0,\alpha\in[0,1]\right.$$

MACLAURIN's Inequality

$$\begin{cases}\frac{a+b+c}{3}\geq\sqrt{\frac{ab+ac+ca}{3}}\geq\sqrt[3]{abc}\\ \frac{a+b+c+d}{4}\geq\sqrt{\frac{ab+ac+ad+bc+bd+cd}{6}}\geq\sqrt[3]{\frac{abc+abd+bcd+acd}{4}}\geq\\ \geq\sqrt[4]{abcd}\\ \frac{\sum x_i}{n}\geq\sqrt{\frac{\sum x_ix_j}{\binom{n}{2}}}\geq\sqrt[3]{\frac{\sum x_ix_jx_k}{\binom{n}{3}}}\geq\cdots\geq\sqrt[n-1]{\frac{\sum x_{i_1}x_{i_2}\cdot\ldots\cdot x_{i_{n-1}}}{\binom{n}{n-1}}}\geq\sqrt[n]{x_{i_1}x_{i_2}\cdot\ldots\cdot x_{i_n}}\end{cases}$$

$$\begin{cases}S_k^2\geq S_{k-1}S_{k+1};\ \sqrt[k]{S_k}\geq\sqrt[k+1]{S_{k+1}}; 1\leq k<n\\ S_k=\frac{1}{\binom{n}{k}}\cdot\sum_{1\leq i_1<\cdots<i_k\leq n}a_{i_1}a_{i_2}\cdot\ldots\cdot a_{i_k}; a_i\geq 0\end{cases}$$

BERNOULLI's Inequality

$$(1+x)^r\geq 1+rx; x\geq -1; r\in(-\infty,0]\cup[1,\infty)$$
$$(1+x)^r\leq 1+rx; x\geq -1; r\in[0,1]$$
$$(1+x)^r\leq 1+(2^r-1)x; x\in[0,1]; r\in(-\infty,0]\cup[1,\infty)$$
$$(1+x)^n\leq\frac{1}{1-nx}; x\in[-1,0];\ n\in\mathbb{N}$$
$$(1+x)^r\leq 1+\frac{rx}{1-(r-1)x}; x\in\left[-1,\frac{1}{r-1}\right];\ r>1$$
$$(1+nx)^{n+1}\geq(1+(n+1)x)^n; x\in\mathbb{R}; n\in\mathbb{N}$$

$$(a+b)^n \le a^n + nb(a+b)^{n-1}; a,b \ge 0; n \in \mathbb{N}$$

$$\left(1+\frac{x}{p}\right)^p \ge \left(1+\frac{x}{q}\right)^q \text{ for } \begin{cases} (i) x > 0; p > q > 0 \\ (ii) -p < -q < x < 0 \\ (iii) -q > -p > x > 0 \end{cases}$$

$$\left(1+\frac{x}{p}\right)^p \le \left(1+\frac{x}{q}\right)^q \text{ for } \begin{cases} (iv) q < 0 < p, -q > x > 0 \\ (v) q < 0 < p, -p < x < 0 \end{cases}$$

TRIGONOMETRIC Inequalities

$$x - \frac{x^3}{2} \le x\cos x \le \frac{x\cos x}{1-x^{\frac{2}{3}}} \le x\sqrt[3]{\cos x} \le x - \frac{x^3}{6} \le x\cos\frac{x}{\sqrt{3}} \le \sin x$$

HIPERBOLIC Inequalities

$$x\cos x \le \frac{x^3}{\sin h^2 x} \le x\cos^2\left(\frac{x}{2}\right) \le \sin x \le (x\cos x + 2x)/3 \le \frac{x^2}{\sin hx}$$

$$\frac{2x}{\pi} \le \sin x \le x\cos\frac{x}{2} \le x \le x + \frac{x^3}{3} \le \tan x; x \in \left[0, \frac{\pi}{2}\right]$$

$$\cos hx + \alpha \sin hx \le e^{x\left(\alpha + \frac{x}{2}\right)}; x \in \mathbb{R}; \alpha \in [-1,1]$$

ACZEL's Inequality

$$\begin{cases} \left(a_1 b_1 - \sum_{i=1}^{n} a_i b_i\right)^2 \ge \left(a_1^2 - \sum_{i=1}^{n} a_i^2\right)\left(b_1^2 - \sum_{i=2}^{n} b_i^2\right) \\ \text{if } a_1^2 > \sum_{i=1}^{n} a_i^2 \text{ or } b_1^2 > \sum_{i=2}^{n} b_i^2 \end{cases}$$

ABEL's Inequality

$$\begin{cases} b_1 \min_k \sum_{i=1}^{k} a_i \le \sum_{i=1}^{n} a_i b_i \le b_1 \max_k \sum_{i=1}^{k} a_i \\ b_1 \ge b_2 \ge \cdots \ge b_n \ge 0; n \in \mathbb{N}^* \end{cases}$$

KY FAN's Inequality

$$\begin{cases} \frac{\prod_{i=1}^{n} x_i^{a_i}}{\prod_{i=1}^{n}(1-x_i)^{a_i}} \le \frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i(1-x_i)}; x_i \in \left[0, \frac{1}{2}\right] \\ a_i \in [0,1]; a_1 + a_2 + \cdots + a_n = 1 \end{cases}$$

SHAPIRO's Inequality

$$\begin{cases} \sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geq \frac{n}{2}; x_i > 0; x_{n+1} = x_1; x_{n+2} = x_2 \\ n \leq 12, n \text{ uneven or } n \leq 23, n \text{ even}; n \in \mathbb{N}^*; n \geq 3 \end{cases}$$

CHONG's Inequality

$$\begin{cases} \sum_{i=1}^{n} \frac{a_i}{a_{\pi(i)}} \geq n; \prod_{i=1}^{n} a_i^{a_i} \geq \prod_{i=1}^{n} a_i^{a_{\pi(a_i)}}; a_i > 0 \\ \pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & i & \cdots & n \\ \cdots & \cdots & \cdots & \pi(i) & \cdots & \cdots & \cdots \end{pmatrix} \in S_n; n \in \mathbb{N}^* \end{cases}$$

SURANYI's Inequality

If $a_1, a_2, \ldots, a_n > 0, n \in \mathbb{N}^*$ then:

$$(n-1)(a_1^n + a_2^n + \cdots + a_n^n) + na_1a_2 \ldots a_n \geq$$
$$\geq (a_1 + a_2 + \cdots + a_n)(a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1})$$

HÖLDER's Reversed Inequality

$$\frac{a_1^p}{b_1^q} + \frac{a_2^p}{b_2^q} + \frac{a_3^p}{b_3^q} \geq \frac{(a_1+a_2+a_3)^p}{(b_1+b_2+b_3)^p} \cdot 3^{1+q-p}$$
$$a_1, a_2, a_3, b_1, b_2, b_3, p, q \in (0, \infty)$$

VIÈTE INTERFERENCES – 1

$$a+b+c = p, ab+bc+ca = q, abc = r$$

$$a^2+b^2+c^2 = p^2-2q$$

$$a^3+b^3+c^3 = p^3-3pq+3r$$

$$a^4+b^4+c^4 = (p^2-2q)^2-2(q^2-2pr)$$

$$a^2b^2+b^2c^2+c^2a^2 = q^2-2pr$$

$$a^2b+ab^2+a^2c+ac^2+b^2c+bc^2 = pq-3r$$

$$(a+b)(b+c)(c+a) = pq-r$$

$$(2a-b-c)(2b-c-a)(2c-a-b) = 2p^3-9pq+27r^2$$

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)=p^2q^2-2p^3q-2p^3+4pqr-r^2$$

$$\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}=\frac{pq-3r}{r}$$

VIÈTE INTERFERENCES – 2

$$a+b+c=p, ab+bc+ca=q, abc=r$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{q}{r}, \qquad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{q^2-2pr}{r^2}$$

$$\left(\frac{a}{b}\right)^2+\left(\frac{a}{c}\right)^2+\left(\frac{b}{a}\right)^2+\left(\frac{b}{c}\right)^2+\left(\frac{c}{a}\right)^2+\left(\frac{c}{b}\right)^2=\frac{p^2q^2-2q^3-2p^3r+4q^2r-3r^2}{r^2}$$

$$\frac{a^2}{b}+\frac{a^2}{c}+\frac{b^2}{a}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{c^2}{b}=\frac{p^2q-2q^2-pr}{r}$$

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}=\frac{q^2-2pr}{r}$$

VIÈTE INTERFERENCES – 3

$$a+b+c=p, ab+bc+ca=q, abc=r$$

$$\frac{(a-b)^2}{a+b}+\frac{(b-c)^2}{b+c}+\frac{(c-a)^2}{c+a}=\frac{2(p^2q-3pr-2q^2)}{pq-r}$$

$$\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\frac{q^3+p^3r-6pqr+9r^2}{r^2}$$

$$(b-a)^2+(c-b)^2+(a-c)^2=2p^2-4q$$

$$(b-a)^3+(c-b)^3+(a-c)^3=2p^3-3pq-3r$$

$$(b-a)^2(c-b)^2(a-c)^2=p^2q^2-4p^3r-4q^3+18pqr-27r^2$$

$$a^5b^2+b^5a^2+a^5c^2+c^5a^2+b^5c^2+c^5b^2=$$
$$=p^3q^2-2p^4r-3pq^3+6p^2qr+3q^2r-7pr^2$$

VIÈTE INTERFERENCES – 4

$$a+b+c=p, ab+bc+ca=q, abc=r$$

$$a^3b^3+b^3c^3+c^3a^3=q^3-3pqr+3r^2$$

$$a^4b^4+b^4c^4+c^4a^4=q^4-4pq^2r+2p^2r^2+4qr^2$$

$$p^2 \geq 3q,\ p^3 \geq 27r, q^2 \geq 3pr,\ pq \geq 9r$$

$$2p^3 + 9r \geq 7pq,\ p^2 + 3pr \geq 4q^2$$

$$p^4 + 4q^2 + 6pr \geq 5p^2q$$

$$r \geq max\left(0, \frac{p(4q - p^2)}{4}, \frac{(4q - p^2)(p^2 - q)}{6p}\right)$$

PQR METHOD – 1

$$a + b + c = p, ab + bc + ca = \frac{p^2 - q^2}{3}, abc = r$$

$$a^2 + b^2 + c^2 = \frac{p^2 + 2q^2}{3}$$

$$a^3 + b^3 + c^3 = pq^2 + 3r$$

$$ab(a + b) + bc(b + c) + ca(c + a) = \frac{p(p^2 - q^2)}{3} - 3r$$

$$(a + b)(b + c)(c + a) = \frac{p(p^2 - q^2)^2}{3} - r$$

$$ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) = \frac{(p^2 + 2q^2)(p^2 - q^2)}{9} - pr$$

$$a^4 + b^4 + c^4 = \frac{-p^4 + 8p^2q^2 + 2q^4}{9} + 4pr$$

PQR METHOD – 2

If

$$a + b + c = p, ab + bc + ca = \frac{p^2 - q^2}{3}, abc = r, q \geq 0$$

then:

$$\frac{(p + q)^2(p - 2q)}{27} \leq r \leq \frac{(p - q)^2(p + 2q)}{27}$$

If $a + b + c = 1$ then:

$$\frac{(1 + q)^2(1 - 2q)}{27} \leq r \leq \frac{(1 - q)^2(1 + 2q)}{27}$$

If $abc = 1$ then:

$$\frac{(p + q)^2(p - 2q)}{27} \leq 1 \leq \frac{(p - q)^2(p + 2q)}{27}$$

EXPONENTIAL Inequalities

$$e^x \geq \left(1+\frac{x}{n}\right)^n \geq 1+x; \left(1+\frac{x}{n}\right)^n \geq e^x\left(1-\frac{x^2}{n}\right);\ n>1; |x| \leq n$$

$$e^x \geq x^e; (\forall)x \in \mathbb{R}$$

$$\frac{x^n}{n!}+1 \leq e^x \leq \left(1+\frac{x}{n}\right)^{n+\frac{x}{2}}; x, n \in (0, \infty)$$

$$e^x \geq 1+x+\frac{x^2}{2}; x \geq 0; e^x \leq 1+x+\frac{x^2}{2}; x \leq 0$$

$$e^{-x} \leq 1-\frac{x}{2}; x \in [0, 1{,}59], 2^{-x} \leq 1-\frac{x}{2}; x \in [0, 1]$$

$$\frac{1}{2-x} < x^x < x^2-x+1; x \in (0, 1)$$

$$x^{\frac{1}{r}}(x-1) \leq rx\left(x^{\frac{1}{r}}-1\right);\ x, r \in [1, \infty)$$

$$x^y+y^x > 1; e^x > \left(1+\frac{x}{y}\right)^y > e^{\frac{xy}{x+y}}; x, y \in (0, \infty)$$

$$2-y-e^{-x-y} \leq 1+x \leq y+e^{x-y}; x, y \in \mathbb{R}$$

$$e^x \leq x+e^{x^2}; x, y \in \mathbb{R}$$

LOGARITHMIC Inequalities

$$\frac{x-1}{x} \leq \ln x \leq \frac{x^2-1}{2x} \leq x-1; \ln x \leq n\left(x^{\frac{1}{n}}-1\right); x, n \in (0, \infty)$$

$$\frac{2x}{1+x} \leq \ln(1+x) \leq \frac{x}{\sqrt{x+1}}; x \geq 0$$

$$\frac{2x}{1+x} \geq \ln(1+x) \geq \frac{x}{\sqrt{x+1}}; x \in (-1{,}0]$$

$$\ln(n+1) < \frac{1}{n}+\ln n \leq \sum_{i=1}^{n}\frac{1}{i} \leq 1+\ln n$$

$$\ln(1+x) \geq \frac{x}{2}; x \in [0, 2{,}51], \ln(1+x) \leq \frac{x}{2}; x \in (-1{,}0] \cup (2{,}51, \infty)$$

$$\ln(1+x) \geq x-\frac{x^2}{2}+\frac{x^3}{4}; x \in [0; 0{,}45]$$

$$\ln(1+x) \leq x-\frac{x^2}{2}+\frac{x^3}{4}; x \in (-\infty, 0) \cup (0{,}45; \infty)$$

$$\ln(1-x) \geq -x-\frac{x^2}{2}-\frac{x^3}{2}; x \in [0{,}0{,}43]$$

$$\ln(1-x) \leq -x-\frac{x^2}{2}-\frac{x^3}{2}; x \in (-\infty, 0) \cup (0{,}43; 1)$$

BINOMIAL Inequalities

$$\max\left\{\frac{n^k}{k^k};\frac{(n-k+1)^k}{k!}\right\} \le \binom{n}{k} \le \frac{n^k}{k!} \le \frac{(en)^k}{k^k}$$

$$\binom{n}{k} \le \frac{n^n}{k^k(n-k)^{n-k}} \le 2^n, \frac{n^k}{4k!} \le \binom{n}{k}$$

$$\text{for } \sqrt{n} \ge k \ge 0$$

$$\frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{8n}\right) \le \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{9n}\right)$$

$$\binom{n_1}{k_1}\binom{n_2}{k_2} \le \binom{n_1+n_2}{k_1+k_2} \text{ for } n_1 \ge k_1 \ge 0; n_2 \ge k_2 \ge 0$$

$$\frac{\sqrt{\pi}}{2}G \le \binom{n}{\alpha n} \le G; G = \frac{2^n H(\alpha)}{\sqrt{2\pi n\alpha(1-\alpha)}};$$

$$H(x) = -\log_2(x^x(1-x)^{1-x})$$

$$\sum_{i=1}^{d}\binom{n}{i} \le n^d + 1;\ \sum_{i=0}^{d}\binom{n}{i} \le 2^n; n \ge d \ge 0$$

$$\sum_{i=0}^{d}\binom{n}{i} \le \left(\frac{en}{d}\right)^d; n \ge d \ge 1$$

$$\sum_{i=0}^{d}\binom{n}{i} \le \binom{n}{d}\left(1+\frac{d}{n-2d+1}\right); \frac{n}{2} \ge d \ge 0$$

$$\binom{n}{\alpha n} \le \sum_{i=0}^{\alpha n}\binom{n}{i} \le \frac{1-\alpha}{1-2\alpha}\binom{n}{\alpha n}; \alpha \in \left(0,\frac{1}{2}\right)$$

GIBBS's Inequality

$$\begin{cases} \sum_{i=1}^{n} a_i \ln\left(\frac{a_i}{b_i}\right) \ge a\ln\left(\frac{a}{b}\right); a_i > 0; b_i > 0 \\ i \in \overline{1,n}; n \in \mathbb{N}^* \\ \sum_{i=1}^{n} a_i f\left(\frac{b_i}{a_i}\right) \le af\left(\frac{b}{a}\right); f \text{ concave}; \end{cases}$$

$$a = a_1 + a_2 + \cdots + a_n; b = b_1 + b_2 + \cdots + b_n$$

KANTOROVICH's Inequality

$$\begin{cases} \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \leq \frac{(m+M)^2}{4mM}\left(\sum_{i=1}^{n} x_i y_i\right)^2 \\ x_i, y_i \in (0,\infty);\ 0 < m \leq \frac{x_i}{y_i} \leq M < \infty \\ i \in \overline{1,n}; n \in \mathbb{N}^* \end{cases}$$

KARAMATA's Inequality

$$\begin{cases} a_1 \geq b_1; a_1 + a_2 \geq b_1 + b_2; a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1} \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n; f \text{ convex} \\ f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n) \end{cases}$$

MURRAY–KLAMKIN Inequality

$$\begin{cases} \frac{\prod_{i=1}^{n}(1+a_i)}{(1+n)^n} \geq \frac{\prod_{i=1}^{n}(1-a_i)}{(n-1)^n}; n \geq 2 \\ a_i \in (0,\infty); i \in \overline{1,n}; a_1 + a_2 + \cdots + a_n = 1; n \in \mathbb{N}^* \end{cases}$$

KURLIANCIK's Inequality

$$\sum_{i=1}^{n} \frac{i}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_i}} < 2\sum_{i=1}^{n} a_i\,; a_i > 0$$

TIBERIU POPOVICIU's Inequality

$$\frac{1}{3}\big(f(x) + f(y) + f(z)\big) + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]$$

$$f: [a,b] \to \mathbb{R}; f \text{ convex}$$

DIAZ–METCALF Inequality

$$\sum_{k=1}^{n} b_k^2 + mM\sum_{k=1}^{n} a_k^2 \leq (m+M)\sum_{k=1}^{n} a_k b_k$$

$$a_k, b_k \in \mathbb{R}^*;\ m \leq \frac{a_k}{b_k} \leq M; k \in \overline{1,n}; n \in \mathbb{N}^*$$

GREUB–RHEIBOLDT Inequality

$$[a,A]; [b,B] \subset (0,\infty);\ x_k \in [a,A];\ y_k \in [b,B]; t_k \in \mathbb{R}; k \in \overline{1,n}; n \in \mathbb{N}^*$$

$$\left(\sum_{k=1}^{n} t_k x_k^2\right)\left(\sum_{k=1}^{n} t_k y_k^2\right) \leq \frac{(ab+AB)^2}{4abAB}\left(\sum_{k=1}^{n} t_k x_k y_k\right)^2$$

POLYA–SZEGO Inequality

$[a,A];[b,B] \subset (0,\infty);\ x_k \in [a,A];\ y_k \in [b,B]; t_k \in \mathbb{R}; k \in \overline{1,n}; n \in \mathbb{N}^*$

$$\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) \leq \frac{(ab+AB)^2}{4abAB}\left(\sum_{k=1}^{n} x_k y_k\right)^2$$

SCHWEITZER's Inequality

$[m,M] \subset (0,\infty);\ x_k \in [m,M];\ k \in \overline{1,n}; n \in \mathbb{N}^*$

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \leq \frac{(m+M)^2}{4mM} n^2$$

HLAWKA's Inequality

$$|x+y+z| + |x| + |y| + |z| \geq |x+y| + |y+z| + |z+x|; x,y,z \in \mathbb{C}$$

ABEL's Inequality

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (a_i - a_{i+1})(b_1 + b_2 + \cdots + b_i)$$

BECKENBACH's Inequality

$$\frac{\sum_{k=1}^{n}(x_k+y_k)^2}{\sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k} \leq \frac{\sum_{k=1}^{n} x_k^2}{\sum_{k=1}^{n} x_k} + \frac{\sum_{k=1}^{n} y_k^2}{\sum_{k=1}^{n} y_k}$$

$$x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n > 0; n \in \mathbb{N}^*$$

USEFUL INEQUALITIES

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}}; n \geq 2;\ \sqrt[n]{n!} > \frac{n}{e}; n \geq 1$$

$$\frac{2}{\sqrt[k]{n+1} + \sqrt[k]{n-1}} > \frac{1}{\sqrt[k]{n}}; k \geq 2; n \geq 1$$

$$\frac{\tan x}{x} > \frac{x}{\sin x}; x \in \left(0, \frac{\pi}{2}\right);\ \sin x + \tan x > 2x; x \in \left(0, \frac{\pi}{2}\right)$$

$$\frac{\pi}{2} - x \geq \cos x\,; x \in \left(0, \frac{\pi}{2}\right), \cos x \geq 1 - \frac{x^2}{2}; x \in \mathbb{R}$$

$$x - \sin x \leq x^3; x \in \left(0, \frac{\pi}{2}\right), \tan x - x \leq x^2; x \in [0,1)$$

$$\frac{x^2 + 2}{\sqrt{x^2 + 1}} \geq 2; x \in \mathbb{R}; \frac{x^2}{1 + x^4} \leq \frac{1}{2}; x \in \mathbb{R}$$

$$2\sqrt{x + y} \geq \sqrt{x} + \sqrt{y}; x + y \geq \sqrt{\frac{x^2 + y^2}{2}} + \sqrt{xy}$$

$$\frac{x}{x^2 + yz} \leq \frac{1}{4}\left(\frac{1}{y} + \frac{1}{z}\right); \frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} \geq \sqrt{x} + \sqrt{y}$$

$$\frac{x + y}{4} + \frac{xy}{x + y} - \sqrt{xy} \leq \frac{x - y}{2}; x \geq y > 0;\ \frac{\sqrt{x - 1}}{x} \leq \frac{1}{2}; x \geq 1;$$

$$x + y \geq \sqrt{\frac{x^2 + y^2}{2}} + \sqrt{xy}; x, y \geq 0$$

$$x^3 + y^3 + z^3 \geq 3xyz; x, y, z \in [0, \infty)$$

$$\frac{1}{(1 + x)^2} + \frac{1}{(1 + y)^2} \geq \frac{1}{1 + xy}; x, y \in (0, \infty);\ xy \geq 1$$

$$\frac{x^3 + y^3}{x^2 + xy + y^2} \geq \frac{x + y}{3}; x, y \in (0, \infty)$$

$$\frac{x^3}{x^2 + xy + y^2} \geq \frac{2x - y}{3}; x, y \in (0, \infty)$$

$$\sqrt{x^2 + xy + y^2} \geq \frac{\sqrt{3}}{2}(x + y);\ x, y \in [0, \infty)$$

$$\sqrt{x^2 + xy + y^2} \leq \sqrt{3(x^2 - xy + y^2)}; x, y \in \mathbb{R}$$

$$x^4 + y^4 \leq \frac{x^6}{y^2} + \frac{y^6}{x^2}; x, y \in \mathbb{R}^*$$

$$\frac{\sqrt{x^2 + 1}}{x^2 + 2} \leq \frac{1}{2}; x \in \mathbb{R}$$

$$\frac{3(x^4+y^4)}{2(x^2+xy+y^2)} \geq x^2-xy+y^2 \geq xy; x,y \in \mathbb{R}$$

$$\frac{\sqrt{x}}{x+1} < \frac{1}{\sqrt{x+2}}; x \in \mathbb{N}^*, \sqrt{\frac{x-1}{x}} < \frac{2x-1}{x}; x > 2$$

$$1 < \sqrt[k]{k} \leq \sqrt[3]{3}; k \geq 4,\ e^x+e^y+\frac{1}{e^{x+y}} \geq 3; x,y \in \mathbb{R}$$

$$\sqrt{x}+\sqrt{y} \leq 2\sqrt{\frac{x+y}{2}}; x,y \geq 0, n^n \leq (n!)^2; n \in \mathbb{N}^*$$

$$\left(\frac{x}{y}\right)^{z-t} \geq 1; x,y,z,t \in \mathbb{R}^*$$

$$\tan x \geq x \geq \sin x \geq \frac{x}{x+1}; x \in \left[0,\frac{\pi}{2}\right)$$

$$(x+y)(x+z) \geq x\left(\sqrt{y}+\sqrt{z}\right)^2; x,y,z \in [0,\infty), e^x \leq x+e^{x^2}; x \in \mathbb{R}$$

$$\frac{x^2+y^2}{x+y} \geq \frac{x+y}{2}; x,y \in \mathbb{R}; x+y \neq 0$$

SCHWEITZER's Inequality – Integral Form

$a,b,m,M \in (0,\infty);\ a<b; m<M, f:[a,b] \to [m,M]; f$ continous function

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b \frac{1}{f(x)}\,dx\right) \leq (m+M)^2(b-a)^2$$

KANTOROVICH's Inequality – Integral Form

$a,b,m,M \in (0,\infty);\ a<b; m<M$
$f,g:[a,b] \to [m,M]; h:[a,b] \to \mathbb{R},\ f,g,h$ continuous functions

$$\left(\int_a^b (f \cdot h)(x)dx\right)\left(\int_a^b \left(\frac{h}{g}\right)(x)dx\right) \leq \frac{(m+M)^2}{4mM}\left(\int_a^b h(x)\,dx\right)^2$$

GREUB–RHEINBOLDT Inequality – Integral Form

$a,b,m_1,M_1,m_2,M_2 \in (0,\infty);\ a<b; m_1<M_1; m_2<M_2$
$f:[a,b] \to [m_1,M_1]; g:[a,b] \to [m_2,M_2], h:[a,b] \to \mathbb{R};$
f,g,h continuous functions

$$\left(\int_a^b (f^2h)(x)dx\right)\left(\int_a^b (g^2h)(x)dx\right) \le \frac{(m_1m_2+M_1M_2)^4}{4m_1m_2M_2M_2}\left(\int_a^b (fgh)(x)\,dx\right)^2$$

PÓLYA–SZEGŐ Inequality – Integral Form

$$a,b,m_1,M_1,m_2,M_2 \in (0,\infty);\ a<b; m_1<M_1; m_2<M_2$$
$$f:[a,b]\to[m_1,M_1]; g:[a,b]\to[m_2,M_2]$$

f,g continuous functions

$$\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right) \le \frac{(m_1m_2+M_1M_2)^2}{4m_1m_2M_1M_2}\left(\int_a^b (fg)(x)dx\right)^2$$

MEANS Inequality – Integral Form

$$m \le \frac{b-a}{\int_a^b \frac{dx}{f(x)}} \le e^{\frac{1}{b-a}\int_a^b \ln f(x)dx} \le \frac{1}{b-a}\int_a^b f(x)\,dx;\ f:[a,b]\to[m,M]$$

CAUCHY–SCHWARZ Inequality – Integral Form

$$\int_a^b |f(x)g(x)|\,dx \le \sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}, f,g:[a,b]\to\mathbb{R}$$

HÖLDER's Inequality – Integral Form

$$\int_a^b |f(x)g(x)|\,dx \le \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}, p>1; \frac{1}{p}+\frac{1}{q}=1$$

MINKOWSKI's Inequality – Integral Form

$$\sqrt{\int_a^b [f(x)+g(x)]^2dx} \le \sqrt{\int_a^b f^2(x)\,dx} + \sqrt{\int_a^b g^2(x)\,dx}$$

$$\left(\int_a^b |f(x)+g(x)|^p dx\right)^{\frac{1}{p}} \le \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p\,dx\right)^{\frac{1}{p}}$$
$$p\ge 1; f,g:[a,b]\to\mathbb{R}$$

CHEBYSHEV's Inequality – Integral Form

$$\begin{cases} f, g: [a, b] \to \mathbb{R} \text{ monotone to the contrary} \\ \int_a^b f(x)g(x)dx \leq \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)\,dx\right) \end{cases}$$

$$\begin{cases} f, g: [a, b] \to \mathbb{R} \text{ monotone in the same sense} \\ \int_a^b f(x)g(x)\,dx \geq \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right) \end{cases}$$

JENSEN's Inequality – Integral Form

$$\begin{cases} f: [a, b] \to [m, M];\ \varphi: [m, M] \to \mathbb{R} \quad \varphi - \text{convex} \\ \varphi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b \varphi\big(f(x)\big)\,dx \end{cases}$$

YOUNG's Inequality – Integral Form

$$\begin{cases} f: (0, \infty) \to (0, \infty) \text{ continuous and increasing} \\ a \in (0, \infty); b \in f\big((0, \infty)\big); f(0) = 0 \\ ab \leq \int_a^b f(x)\,dx + \int_0^b f^{-1}(y)dy \end{cases}$$

CAUCHY's Inequality – Integral Form

$f: [a, b] \to \mathbb{R}; f$ increasing, $f(a) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f(b)$

HERMITE–HADAMARD Inequality – Integral Form

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

$f: [a, b] \to \mathbb{R}; f \in C^2([a, b]); f$ convex