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OLYMPIAD PROBLEMS FROM ALL OVER THE WORLD

VOLUME 3
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Cartea Românească
EDUCAȚIONAL

Dedicated to the International Mathematical Olympiad

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Chapter I
Problems

1. Let $ABCDE$ be a regular pentagon with center M. A point $P \neq M$ is chosen on the line segment MD. The circumcircle of ABP intersects the line segment AE in A and Q and the line through P perpendicular to CD in P and R.

Prove that AR and QR are of the same length.

STEPHAN WAGNER, AUSTRIAN NMO, 2017

2. Let ABC be an acute triangle. Let H denote its orthocenter and D, E and F the feet of its altitudes from A, B and C, respectively. Let the common point of DF and the altitude through B be P. The line perpendicular to BC through P intersects AB in Q. Furthermore, EQ intersects the altitude through A in N.

Prove that N is the mid-point of AH.

KARL CZAKLER, AUSTRIAN NMO, 2017

3. The diagonals AC and BD of the convex quadrilateral $ABCD$ intersect at point O. The points A_1, B_1, C_1 and D_1 from the segments AO, BO, CO and DO, respectively, are such that $AA_1 = CC_1$ and $BB_1 = DD_1$. Let M be the second intersection point of the circumcircles of ΔAOB and ΔCOD; N be the second intersection point of circumcircles of ΔAOD and ΔBOC; P be the second intersection points of the circumcircles of ΔA_1OB_1 and ΔC_1OD_1 and Q be the second intersecting point of circumcircles of ΔA_1OD_1 and ΔB_1OC_1. Prove that the points M, N, P and Q are concyclic.

ALEKSANDAR IVANOV, BULGARIAN NMO, 2017

4. Consider acute scalene ΔABC with altitudes CD, AE and BF. The points E' and F' are symmetric to E and F with respect to A and B, respectively. Point C_1 on the ray $\overrightarrow{CD}$ is such that $DC_1 = 3CD$. Prove that $\sphericalangle E'C_1F' = \sphericalangle ACB$.

STANISLAV CHOBANOV, BULGARIAN NMO, 2017

5. A square is cut into several rectangles, none of which is a square, so that the sides of each rectangle are parallel to the sides of the square. For each rectangle with sides a, b, $a < b$, compute the ratio $\frac{a}{b}$. Prove that sum of these ratios is at least 1.

SINGAPORE SMO, 2017

6. In ΔABC, $AB = AC$, D is a point on the side BC and E is a point on the segment AD. Given that $\sphericalangle BED = \sphericalangle BAC = 2\sphericalangle CBD$, prove that $BD = 2CD$.

SINGAPORE SMO, 2017

7. Let a, b, c be nonzero integers, with 1 as their only positive common divisor, such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. Find the number of such triples (a, b, c) with:

$$50 \geq |a| \geq |b| \geq |c| \geq 1.$$

SINGAPORE SMO, 2017

8. In the cyclic quadrilateral $ABCD$, the sides AB, DC meet at Q, the side AD, BC meet at P, M is midpoint of BD. If $\measuredangle APQ = 90°$, prove that PM is perpendicular to AB.

SINGAPORE SMO, 2017

9. The incircle of ΔABC touches the sides BC, CA, AB at D, E, F, respectively. A circle through A and B encloses ΔABC and intersects the line DE at points P and Q. Prove that the midpoint of AB lies on the circumcircle of ΔPQF.

SINGAPORE SMO, 2017

10. The four digit number $ABCD$ has the property that:

$$ABCD = A \cdot BCD + ABC \cdot D.$$

What is the smallest possible value of $ABCD$?

GORDON LESSELS, IRELAND SHL, 2017

11. The images of the reflection of the circumcentre of triangle ABC in the sides of the triangle are X, Y and Z. Prove that XYZ is congruent to the triangle ABC and corresponding sides are parallel.

JIM LEAHY, IRELAND SHL, 2017

12. Given a point P between the legs of an angle with vertex A. Show, with proof, how to construct a line trough P that intersects the legs of the angle at points B and C so that $|PB| = |PC|$.

JIM LEAHY, IRELAND SHL, 2017

13. Two circles intersect at A and B. A common tangent is drawn to the circles at P and Q. A circle is drawn through P, Q and A and the line AB meets this circle again at C. Join CP and CQ and extend both to meet the given circles at F and B, respectively. Prove that P, Q, F and E lie on the circumference of a circle.

JIM LEAHY, IRELAND SHL, 2017

14. Let O be the circumcenter of an acute triangle ABC and let O_1 and O_2 be circumcenters of triangles OAB and OAC, respectively. The circumcircles of triangles OAB and OAC intersect BC at D ($\neq B$) and E ($\neq C$), respectively and the perpendicular bisector of BC intersects AC at F ($\neq A$). Show that the circumcenter of the triangle ADE lies on AC if and only if the point F lies on the line passing O_1 and O_2.

KOREAN NMO, 2017

15. Let L, M be the midpoints of two sides AB, CD of a convex cyclic quadrilateral $ABCD$. Let E be the intersection of its diagonals AC and BD. Suppose the rays AB and DC intersect at a point F and LM and DE intersect at a point P. Let Q be the foot of the perpendicular on the line segment EM from P. Show that if E is the orthocenter of the triangle FLM, then:

$$\frac{EP^2}{EQ}=\frac{1}{2}\left(\frac{BD^2}{DF}-\frac{BC^2}{CF}\right).$$

KOREAN NMO, 2017

16. Four circles are drawn with the sides of the quadrilateral $ABCD$ as diameters. The two circles passing through A meet again at A', the two circles through B at B', the two circles through C at C' and the two circles through D at D'. Suppose that the points A', B', C' and D' are distinct. Prove that the quadrilateral $A'B'C'D'$ is similar to the quadrilateral $ABCD$.
(*Note:* Two quadrilaterals are *similar* if their corresponding angles are equal to each other *and* their corresponding side lengths are in proportion to each other.)

JIM LEAHY, IRELAND NMO, 2017

17. A line segment B_0B_n is divided into n equal parts at points $B_1, B_2, \ldots, B_{n-1}$ and A is a point such that $\sphericalangle B_0AB_n$ is a right angle. Prove that:

$$\sum_{k=0}^{n}\left|AB_k\right|^2=\sum_{k=0}^{n}\left|B_0B_k\right|^2 .$$

JIM LEAHY, IRELAND NMO, 2017

18. Jake has 99 empty bags. An unlimited supply of balls is available, where the weight or each ball is a non-negative integer power of 3. Jake chooses a finite number of balls and distributes them into the bags such that each bag contains the same total weight. If, no matter how the bags have been filled, Jake must have chosen k balls of the same weight, find the largest possible value of k.

MARK FLANAGAN, IRELAND NMO, 2017

19. If $a, b, c \geq 0$, $a+b+c=ab+be+ca$, then:

$$\frac{(1+a)(1+b)}{3(1+a^2)}+\frac{(1+b)(1+c)}{3(1+b^2)}+\frac{(1+c)(1+a)}{3(1+c^2)}\leq 1+a+b+c .$$

DANIEL SITARU, RMM, ROMANIA

20. If $a, b, c, d > 0$, $a^2+b^2+c^2+d^2=4$, then:

$$\sum\frac{(2a+b+c)(2b+c+d)(2c+d+a)(2d+a+b)}{(a+b+c+d)^3}\leq 16 .$$

DANIEL SITARU, RMM, ROMANIA

21. If a, b, $c > 0$, then:

$$\left(\sum a^2b^2\right)\left(\sum a^4b^4\right)\left(\sum \frac{1}{a^2b^2}\right)\left(\sum \frac{1}{a^4b^4}\right) \geq \left(\sum a\right)\left(\sum a^2\right)\left(\sum \frac{1}{a}\right)\left(\sum \frac{1}{a^2}\right).$$

DANIEL SITARU, RMM, ROMANIA

22. If x, y, $z > 0$, then:

$$\frac{(x+y+z)\sum (x+y)^2 + 2(x+y)(y+z)(z+x)}{4(x+y+z)^3} \geq \frac{13}{27}.$$

DANIEL SITARU, RMM, ROMANIA

23. Prove that if x, y, $z > 0$, then:

$$\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}} \leq \sqrt{6\left(\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x}\right)}.$$

DANIEL SITARU, RMM, ROMANIA

24. If a, b, $c > 0$, $a^2 + b^2 + c^2 = 3$, then:

$$\sum a(a+1)(a+2)(a+3) \geq 72.$$

DANIEL SITARU, RMM, ROMANIA

25. If a, b, c, d, $e > 0$, $a + b + c + d + e > 5$, then:

$$\sum \frac{b+c+d+e}{2a+b+c+d+e} \geq \frac{10}{3}.$$

DANIEL SITARU, RMM, ROMANIA

26. If a, b, $c > 0$, then:

$$\left(4\sqrt{\frac{a}{b}} - \frac{a^2}{b^2}\right) + \left(4\sqrt{\frac{b}{c}} - \frac{b^2}{c^2}\right) + \left(4\sqrt{\frac{c}{a}} - \frac{c^2}{a^2}\right) \leq 9.$$

DANIEL SITARU, RMM, ROMANIA

27. If x, y, $z > 0$; $x + y + z = 1$, then:

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} + 2\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) \geq 3.$$

DANIEL SITARU, RMM, ROMANIA

28. If x, y, $z > 0$; $x + y + z = 1$, then:

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} + \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z} \geq 7.$$

DANIEL SITARU, RMM, ROMANIA

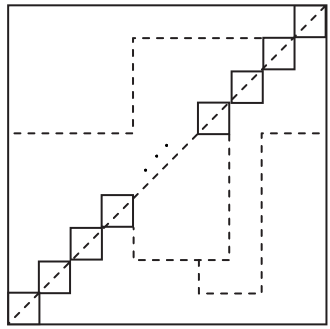
29. Son, his Dad and his Grandfather has ran from their home to a shop and back. Son's velocity was constant, Grandfather's velocity was two times greater than Son's while he was running to the shop and three times less when he was running back. Dad's velocity was two times less than Son's on the way to shop arid 3 times greater when he was running back. Who was the first and who was the last to come home?

UKRAINIAN NMO, 2017

30. There are 22 cards, where the numbers 1, 2, …, 22 are written. Using this cards, one formed 11 fractions. What is the greatest possible number of integer numbers among the fractions?

UKRAINIAN NMO, 2016

31. Given a checked square. One draw a big diagonal and paint black all the cells such that their centers belong to this diagonal. After the cells on the upper side are cut into two pieces and the lower side is cut into three pieces. It occurred that the areas of these figures are 70, 80, 90 and 100. What is the possible area of the last figure?



BOGDAN RUBLYOV, UKRAINIAN NMO, 2016

32. 30 children – boys and girls – formed a circle. It occurred that there is no child such that both its neighbours are boys. What is the least possible number of girls there?

UKRAINIAN NMO, 2017

33. Let ABC be a triangle. Suppose that AD and BE are its angle bisectors. Prove that $\measuredangle ACB = 60°$.

DANILO KHILKO, UKRAINIAN NMO, 2016

34. Is it possible to cut a regular triangle into:

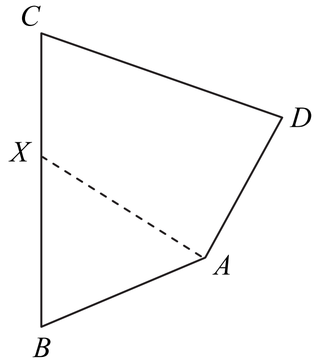
a) three equal quadrilaterals;

b) three equal pentagons?

Convexes of quadrilaterals and pentagons is not obliged.

UKRAINIAN NMO, 2016

35. In the quadrilateral $ABCD$, which is depicted at figure below, the following conditions hold: $\measuredangle ABC = \measuredangle BCD$ and $2AB = CD$. The point X is chosen on the side BC, such that $\measuredangle BAX = \measuredangle CDA$. Prove that $AX = AD$.



UKRAINIAN NMO, 2016

36. Prove that if a, b, x, y, $z \in (0, \infty)$, then:

$$\frac{yz\left(a^2y+b^2z\right)}{x}+\frac{zx\left(a^2z+b^2x\right)}{y}+\frac{xy\left(a^2x+b^2y\right)}{z}\geq\frac{2}{3}ab\left(x+y+z\right)^2.$$

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU,
ROMANIA, RMM SUMMER EDITION, 2016

37. Prove that if $a, b, c > 0$; $a + b + c = 3$, then:

$$\sum a\left(\frac{1}{b^3}+\frac{1}{c^3}\right)\geq\frac{18}{a^3+b^3+c^3}.$$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

38. If a, b, c are the length's sides in any triangle, the following relationship doesn't hold:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{2}{3}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

REDWANE EL MELLAS, MORROCO, RMM SUMMER EDITION, 2016

39. Prove that if $a, b, c \in \mathbb{R}$; $0 < c \leq b \leq a$, then:

$$\left(a+2b\right)\left(a+2c\right)\left(b+2c\right)\leq 8\prod\left(\frac{a^2+ab+b^2}{a+b}\right)\leq\left(2a+b\right)\left(2a+c\right)\left(2b+c\right).$$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

40. Prove that if $a,b,c\in\left(0,\infty\right)$; $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$, then:

$$\frac{a\sqrt{b}+b\sqrt{a}}{a-\sqrt{ab}+b}+\frac{b\sqrt{c}+c\sqrt{b}}{b-\sqrt{bc}+c}+\frac{c\sqrt{a}+a\sqrt{c}}{c-\sqrt{ca}+a}\leq 6.$$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

41. Prove that if a, b, $c \in (0, \infty)$, then:

$$12\sum \frac{c}{a^2+b^2+9} \le \frac{1}{abc}\sum c^2\sqrt{a^2+b^2}\ .$$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

42. The points A_1 and C_1 are chosen on the sides BC and AB of the triangle ABC so that the segments AA_1 and CC_1 are equal and perpendicular. Prove that if $\measuredangle ABC = 45°$, then $AC = AA_1$.

ANDREI GOGOLEV, UKRAINIAN NMO, 2017

43. A point M is chosen on a circle with diameter AB. A point Qi is also taken on this circle such that $\measuredangle MK_iB < 90°$, where K_i is the intersection point of MQ_i and AB. The chord, which is perpendicular to AB and passes through K_i, intersect BQ_i at P_i. Prove that the points P_i belong to a fixed line, while Q_i vary.

IGOR NAGEL, UKRAINIAN NMO, 2016

44. Let AM be a median in an acute triangle ABC. Its extension intersect the circumcircle w of ABC at P. Let AH_1 be an altitude of ΔABC, H – its orthocenter. The rays MH and PH_1 intersect w at K and T respectively. Prove that the circumcircle of ΔKTH_1 is tangent to BC.

KHILKO DANYLO, UKRAINIAN NMO, 2017

45. Let a, b and c be positive real numbers with $a + b + c = 1$. Prove the inequality:

$$a\sqrt{2b+1}+b\sqrt{2c+1}+c\sqrt{2a+1} \le \sqrt{2-(a^2+b^2+c^2)}\ .$$

NIKOLA PETROVIĆ, SERBIAN NMO, 2017

46. Let k be the circumcircle of triangle ABC and let k_a be its excircle opposite to A. The two common tangents of k and k_a meet the line DC at points P and Q. Prove that $\measuredangle PAD = \measuredangle QAC$.

DUŠAN ĐUKIĆ, SERBIAN NMO, 2017

47. Let $ABCD$ be a cyclic quadrilateral. Let O be the circumcenter of the quadrilateral $ABCD$. The diagonals AC and BD intersect at G. Let P, Q, R and S be the circumcenters of triangles AGB, BGC, CGD and DGA respectively. The lines PR and QS intersect at M. Show that M is the midpoint of G and O.

THAILAND NMO, 2017

48. Let ABC be an acute triangle with height AD and $AD = CD$. The median CM intersects it N. Prove that ABC is an isosceles triangle if and only if $CN = 2AM$.

THAILAND NMO, 2017

49. Let ABC be an acute triangle with attitudes AK, BL, CM. Prove that triangle ADC is isosceles if and only if $AM + BK + CL = AL + DM + CK$.

JAROMIR SIMSA, CZECH & SLOVAK NMO, 2017

50. Let D be an arbitrary point on the base AB of an isosceles triangle ABC. Let E be such that $ADEC$ is a parallelogram. Point F on the ray opposite to ED satisfies $EF = EB$. Prove that the length of a chord cut by line BE in the circumcircle of triangle ABE is twice the length of AC.

JAN KUCHARIK, PATRIK BAK, CZECH & SLOVAK NMO, 2017

51. Let ABC be an acute triangle with altitudes BD, CE. Given that $AE \cdot AD = BE \cdot CD$, what is the smallest possible measure of $\measuredangle BAC$?

PATRIK BAK, CZECH & SLOVAK NMO, 2017

52. If $x, y, z > 0$, then:

$$\left(x^2+2\right)\left(y^2+2\right)\left(z^2+2\right) > 16\sqrt{2xyz\sqrt{xyz}}\ .$$

DANIEL SITARU, RMM, ROMANIA

53. Let $h_1, h_2, h_3, m_1, m_2, m_3$ be the altitudes, respectively the medians of intouch triangle in ΔABC. Prove that:

$$\frac{\frac{1}{h_1^2}+\frac{1}{h_2^2}+\frac{1}{h_3^2}}{\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}} = \frac{4R^2}{r^2}\cdot\frac{m_1^2+m_2^2+m_3^2}{m_a^2+m_b^2+m_c^2}.$$

MEHMET SAHIN, RMM, TURKEY

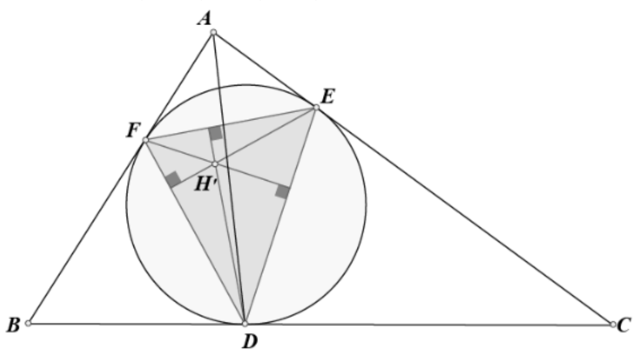
54. Let ΔDEF, $\Delta I_aI_bI_c$ be the contact, respectively the excentral triangle of ΔABC. If $S_1 = S[AFE]$, $S_2 = S[BDF]$, $S_3 = S[CDE]$, $S_0 = S[DEF]$, $A = S[I_aI_bI_c]$, then:

$$4A^2S_1S_2S_3 = S^5,\ S_0A = S^2.$$

MEHMET SAHIN, RMM, TURKEY

55. Prove that:

$$\left(DH'\right)^2+\left(EH'\right)^2+(FH')^2 \geq 3r^2.$$



ABDILKADIR ALTINTAS, RMM, TURKEY

56. If $a,\ b>0,\ a^2+b^2=2$, then:

$$(1+2ab)(2+3ab)(1+3ab)\leq 60(2-ab)^3.$$

DANIEL SITARU, RMM, ROMANIA

57. If in ΔABC the nine-point circle and the circumcenter are tangents, then:

$$abc<\frac{8\sqrt{3}}{3}R^3.$$

DANIEL SITARU, RMM, ROMANIA

58. If $x, y, z > 0$, then:

$$\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x}\leq\sqrt{\frac{3(x+y+z)}{xyz}}.$$

DANIEL SITARU, RMM, ROMANIA

59. Let the internal angle bisector of $\measuredangle BAC$ of ΔABC meet side BC at D. Let Γ be the circle through A tangent to BC at D. Suppose Γ meets sides AB and AC at E and F again, respectively. Lines BF and CE meet Γ again at F and Q, respectively. Let AP and AQ intersect side BC at X and Y, respectively. Prove that:

$$XY=\frac{1}{2}BC.$$

HONG KONG, PREIMO, 2017, MOCK EXAM

60. Find all pairs (x, y) of integers satisfying the equation:

$$x^4-(y+2)x^3+(y-1)x^2+(y^2+2)x+y=2.$$

NGUYEN VIET HUNG, RMM, WINTER EDITION, 2016

61. Find the numbers $a, b, c\in\mathbb{N}^*$, knowing that:

$$\frac{a+1}{b}\in\mathbb{N},\ \frac{b+1}{c}\in\mathbb{N}\ \text{and}\ \frac{c+1}{a}\in\mathbb{N}.$$

GHEORGHE ALEXE, GEORGE-FLORIN ȘERBAN, RMM, WINTER EDITION, 2017

62. Let $ABCD$ be a trapeze, where $AB \parallel CD$; $AB = a$; $CD = b$; $AD = c$; $BC = d$; $a > d$. Prove that:

$$\text{Area }[ABCD]<\frac{(a+b)(a-b+c+d)^2}{16(a-b)}.$$

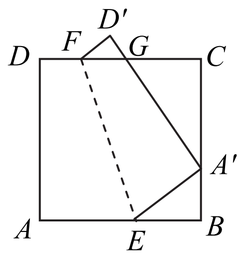
DANIEL SITARU, RMM WINTER EDITION, 2016

63. Let ABC an acute-angled triangle with incentre I. Draw a line to BI at I and let it intersect BC and BA at D and E, respectively. Let P and Q be, respectively, the incentres of the triangles BIA and BIC. Suppose the four points D, E, P, Q are concyclic. Prove that:

$$BA = BC.$$

INDIA, TST, 2017

64. In the given figure, $ABCD$ is a square paper. It is folded along EF such that A goes to a point $A' \neq C$, B on the side DC and D goes to D'. The line $A'D'$ cuts CD in G. Show that the inradius of the triangle GCA' is the sum of the inradii of the triangles $GD'F$ arid $A'BE$.



INDIAN NMO, 2017

65. Let $ABCDE$ be a convex pentagon in which $\measuredangle A = \measuredangle B = \measuredangle C = \measuredangle D = 120°$ and whose side lengths are 5 consecutive integers in some order. Find all possible values of $AB + BC + CD$.

INDIAN NMO, 2017

66. Let ABC be a triangle with $\measuredangle A = 90°$ and $AB < AC$. Let AD be the altitude from A onto BC. Let P, Q and I denote, respectively, the incentres of triangles ABD, ACD and ABC. Prove that:

AI is perpendicular to PQ and $AI = PQ$.

INDIAN NMO, 2017

67. Given $a, b, c > 0$ and $a^2 + b^2 + c^2 = 6$, prove:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + a + b + c \geq 6.$$

NGUYEN PHUC TANG, RMM SPRING EDITION, 2017

68. Let D be the midpoint of side BC of a triangle ABC. Points E and F are taken on the respective sides AC and AB, such that $DE = DF$ and $\measuredangle EDF = \measuredangle BAC$. Prove that:

$$DE \geq \frac{AB + AC}{4}.$$

DUŠAN ĐUKIĆ, SERBIAN TST, 2017

69. Let P be a point inside ΔABC. Let A_1, B_1, C_1 be points in the interiors of the segments PA, PB, PC, respectively. Let $\overline{BC_1} \cap \overline{CB_1} = \{A_2\}$, $\overline{CA_1} \cap \overline{AC_1} = \{B_2\}$ and $\overline{AB_1} \cap \overline{BA_1} = \{C_2\}$. Let U be the intersection of the lines A_1B_1 and A_2B_2 and V be the intersection of the lines A_1C_1 and A_2C_2. Show that the lines UC_2, VB_2 and AP are concurrent.

PAKAWUT JRRADILOK, WIJIT YANGJIT, THAILAND TST, 2017

70. Let D and E be points on sides AB and AC of triangle ABC, respectively, such that $BD = CE$. Let M and N be midpoints of BC and DE, respectively. Prove that MN is parallel to the bisector of $\sphericalangle BAC$.

B. BATTSENGEL, MONGOLIAN TST, 2017

71. Two circles ω_1 and ω_2 intersected at points A and B. Line through B intersects the ω_1 at point C and intersects the ω_2 at point D. The line AD intersects the ω_1 at point E different from A and the line AC intersects the ω_2 at point F different from A. If O is circumcenter of triangle AEF, then prove that $OB \perp CD$.

B. BAT-OD, MONGOLIAN NMO, 2017

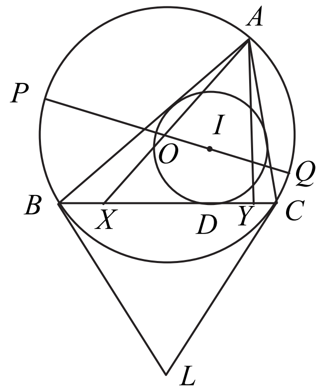
72. Let ω be circumcircle of triangle ABC and let AD and BE be altitudes. A line DE intersects the circle ω at points P and Q with order P, D, Q in the line. Let bisectors of angle APQ and BQP intersect a circle ω at point K and L, respectively. Prove that the line KL is perpendicular to the bisector of angle ACB.

B. ULZIINASAN, NMO, 2017

73. Let ω the circumcircle of a scalene triangle ABC. The tangents to ω at A and C meet in P and the line BP intersects ω in D. Let BB' be a diameter of ω. The exterior angle bisector of $\sphericalangle ABC$ and the lines $B'A$ and $B'C$ intersect in A' and C', respectively. Prove that A', B', C', D are cyclic.

G. BATZAYA, MONGOLIAN NMQ, 2017

74. As shown in the figure, in an acute triangle ABC with $AB > AC$, AY is the altitude from A, $\odot\, O$ and $\odot\, I$ are the circumcircle and incircle of triangle ABC, respectively. $\odot\, I$ touches BC at point D, line AO meets BC at point X. The tangent lines of $\odot\, O$ at points B and C meet at point L. Let PQ be the diameter of $\odot\, O$ passing through I. Show that the points A, D, L are collinear if and only if the points P, X, Y, Q are concyclic.



CHINA NMO, 2017

75. If a and b are positive integers, then $\overline{\overline{a.b}}$ is a decimal number obtained by writing the number a, then the decimal point and then the number b. For example, if $a = 20$ and $b = 17$, then $\overline{\overline{a.b}} = 20.17$ and $\overline{\overline{b.a}} = 17.2$. Determine all pairs (a, b) of positive integers such that $\overline{\overline{a.b}} \cdot \overline{\overline{b.a}} = 13$.

CROATIAN NMO, 2017

76. A triangle ABC is given. Circle k touches $\overline{BC}$ from outside the triangle at point K and the extensions of lines $\overline{AB}$ and $\overline{AC}$ over points B and C at points L and M, respectively. The circle with diameter $\overline{BC}$ intersects segment $\overline{LM}$ at points P and Q so that point P lies between L and Q. Prove that lines BP and CQ intersect at the centre of circle k.

STIPE VIDAK, CROATIAN NMO, 2017

77. Prove that the following inequalities hold for all positive real numbers:

a) $\dfrac{a^3}{ab+c^2}+\dfrac{b^3}{bc+a^2}+\dfrac{c^3}{ca+b^2} \geq \dfrac{3}{2}\cdot\dfrac{a^2+b^2+c^2}{a+b+c}$;

b) $\dfrac{1}{a(b+c)}+\dfrac{1}{b(c+a)}+\dfrac{1}{c(a+b)} \geq \dfrac{3}{2}\cdot\dfrac{a+b+c}{a^3+b^3+c^3}$.

NGUYEN VIET HUNG, RMM WINTER EDITION, 2017

78. Prove that the following inequalities hold for all positive real numbers a, b, c:

a) $\dfrac{b}{a^2}+\dfrac{c}{b^2}+\dfrac{a}{c^2} \geq \dfrac{3(a+b+c)}{a^2+b^2+c^2}$;

b) $\dfrac{b^3}{a^2}+\dfrac{c^3}{b^2}+\dfrac{a^3}{c^2} \geq \dfrac{3(a^2+b^2+c^2)}{a+b+c}$.

NGUYEN VIET HUNG, RMM WINTER EDITION, 2017

79. Prove that for all positive real numbers a, b, c:

$$\frac{a(b^2+c^2)}{2a^2+bc}+\frac{b(c^2+a^2)}{2b^2+ca}+\frac{c(a^2+b^2)}{2c^2+ab}\geq\frac{6abc}{ab+bc+ca}.$$

HUNG NGUYEN VIET, RMM WINTER EDITION, 2017

80. Let $x, y, z > 0$ be positive real numbers. Then:

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\geq\frac{4\sqrt{3xyz(x+y+z)}}{(x+y)(y+z)(z+x)}.$$

D.M. BĂTINEȚU-GIURGIU, MARTIN LUKAREVSKI, RMM WINTER EDITION, 2017

81. Prove that for all positive real numbers a, b, c:

a) $\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}+\frac{(b-c)^2}{2(a+b+c)}$;

b) $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{a+b+c}{2}+\frac{(a+b-2c)^2}{2(a+b+c)}$.

NGUYEN VIET HUNG, RMM WINTER EDITION, 2017

82. Prove that if $x_i\in(0,\infty)$; $i\in\overline{1,n}$; $n\in\mathbb{N}$; $n\geq 3$; $x_{n+1}=x_1; x_1x_2\cdot\ldots\cdot x_n=1$, then:

$$\sum_{i=1}^{n}\frac{\frac{x_i}{x_{i+1}}+\frac{x_{i+1}}{x_i}+1}{\sqrt{x_i^2+x_ix_{i+1}+x_{i+1}^2}}\geq n\sqrt{3}.$$

DANIEL SITARU, RMM WINTER EDITION, 2017

83. (*Juniors*) Juku conjectured the following in his mathematics circle: whenever the product of two coprime integers x and y is divisible by the product of some two coprime integers a and b, at least one of x and y is divisible by a or b. Does his proposition hold?

ESTONIAN NMO, 2017

84. (*Juniors*) Solve the system $a^3 + b = 4c$, $a + b^3 = c$, $ab = -1$.

ESTONIAN NMO, 2017

85. (*Juniors*) Does there exist a positive integer n which has exactly 9 positive divisors and whose all divisors can be placed in a 3-by-3 table such that the products of the 3 numbers in each row, each column and on each diagonal are all the same?

ESTONIAN NMO, 2017

86. (*Juniors*) Juku thought of a 3-digit number that, when reversing the order of the digits, stays the same 3-digit number. Juku noticed that when adding 2016 to that number, the 4-digit number that arises is again the same 4-digit number when reading the digits from right to left. What number did Juku think of?

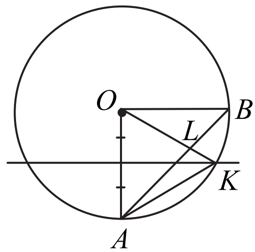
ESTONIAN NMO, 2017

87. (*Juniors*) a) Let a and b be arbitrary positive integers of equal parity. Can we always find noninteger numbers x and y such that $x + y$ and $ax + by$ are integers?

b) The same question when a and b have different parities.

ESTONIAN NMO, 2017

88. (*Juniors*) Let A and B be such points of the circle with centre O that the triangle AOB is right-angled. The perpendicular bisector of the segment AO intersects the shorter arc AB in point K. The lines KO and AB intersect in point L. Prove that the triangle KBL is isosceles.



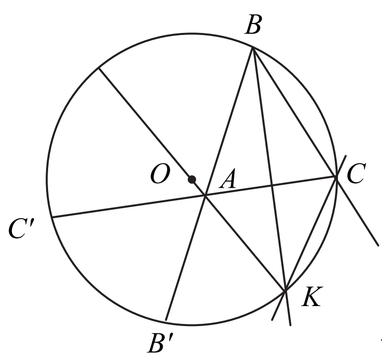
ESTONIAN NMO, 2017

89. (*Seniors*) On the sides BC, CA and AB of triangle ABC, respectively, points D, E and F are chosen. Prove that:

$$\frac{1}{2}(BC + CA + AB) < AD + BE + CF < \frac{3}{2}(BC + CA + AB).$$

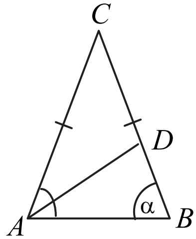
ESTONIAN NMO, 2017

90. (*Seniors*) The bisector of the exterior angle at vertex C of the triangle ABC intersects the bisector of the interior angle at vertex B in point K. Consider the diameter of the circumcircle of the triangle BCK whose one endpoint is K. Prove that A lies on this diameter.



ESTONIAN NMO, 2017

91. Triangle ABC has $AC = BC$. The bisector of angle CAB meets side BC at point D. The difference of the sizes of some two internal angles of triangle ABD is 40°. Find all possibilities of what the size of angle ACB can be.



Estonian NMO, 2017

92. Let ABC be a scalene triangle with median AM. Let K be the point of tangency of de incircle of triangle ABC with the side BC. Prove that if the length of the side BC is the arithmetic mean of the lengths of the sides AB and AC, then the bisector of the angle BAC passes through the midpoint of the segment KM.

Estonian NMO, 2017

93. Solve for natural numbers:

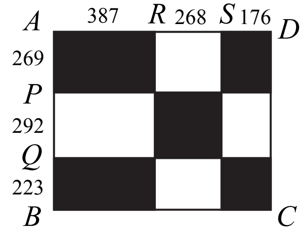
$$\frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\ldots+\frac{10^4}{x+9}=3025.$$

Daniel Sitaru, RMM, Romania

94. For a quadrilateral $ABCD$, $\sphericalangle A = \sphericalangle B = \sphericalangle C = 30°$, $AB = 4$, $BC = 2\sqrt{3}$ are satisfied. Find the area of the quadrilateral $ABCD$. Here, by XY we denote the length of the line segment XY.

Japan NMO, 2017

95. As indicated in the diagram below, a rectangle $ABCD$ is partitioned into 9 sub-regions by drawing 4 line segments parallel to the sides of the rectangle and they are colored black and white alternatingly. Suppose $AP = 269$, $PQ = 292$, $QB = 223$, $AR = 387$, $RS = 263$, $SD = 176$. Find the value of the area of the region colored black minus the area of the region colored white. Here by XY, we denote the length of the line segment XY.



Japan NMO, 2017

96. 3 distinct points D, B, C lie on a common straight line and they satisfy $DB - BC = 2$. Suppose that the point A satisfies the condition $AB = AC$ and, furthermore, that there exists a circle Γ which is tangent to the lines AC and DC at A and D, respectively. Let E be the point of intersection of Γ and the line AB, different from A and let F be the point of intersection of Γ and the line CE, different from E. Find the length of the line segment EF. Here, we denote by XY the length of the line segment XY.

JAPAN NMO, 2017

97. 2 points D, E lie on the side BC of a triangle ABC. 4 points B, D, E, C are located on the side BC in this order and $\measuredangle BAD = \measuredangle ACE$, $\measuredangle ABD = \measuredangle CAE$ are satisfied. Let X be the point of intersection, different from A, of the circumcircle of the triangle ABE and the circumcircle of the triangle ADC. Let F be the point of intersection of the lines AX and BC. Find the length of the line segment XE if $BF = 5$, $CF = 6$, $XD = 3$. Here, we denote by PQ the length of the line segment PQ.

JAPAN NMO, 2017

98. For a triangle ABC, $\measuredangle B = 90°$, $AB = 8$, $BC = 3$ are satisfied. For points P on the side BC, Q on the side CA and R on the side AB, conditions $\measuredangle CRP = \measuredangle CRQ$, $\measuredangle BPQ = \measuredangle CPQ$ are satisfied. It is also known that the length of the circumference of the triangle PQR is 12. Find the length of the perpendicular line drawn from Q to the side BC. Here, we denote by XY the length of the line segment XY.

JAPAN NMO, 2017

99. Find $x, y, z \in \mathbb{N}^*$ such that:

$$\sqrt{\overline{\underbrace{xxxx\ldots xx}_{\text{for 2000 times}}} - \overline{\underbrace{yyyy\ldots yy}_{\text{for 1000 times}}}} = \overline{\underbrace{zzzz\ldots zz}_{\text{for 1000 times}}}$$

DANIEL SITARU, RMM, ROMANIA

100. Let ABC be an acute triangle and O be the circumcenter. Let D, E, F be the foot of the perpendicular line drawn from the vertex A, B, C, respectively, to the opposite side of the triangle. Let M be the midpoint of the side BC. Let X be the point of intersection of the lines AD and EF, Y be the point of intersection of the lines AO and BC and Z be the mid-point of the line segment XY. Prove that 3 points, A, Z, M, are collinear.

JAPAN NMO, 2017

101. Let ABC be a triangle with $\measuredangle A < \measuredangle B < 90°$ and let Γ be the circle through A, B and C. The tangents to Γ at A and C meet at P. The line segments AB and PC produced meet at Q. It is given that:

$$[ACP] = [ABC] = [BQC].$$

Prove that $\measuredangle BCA = 90°$. Here, $[XYZ]$ denotes the area of triangle XYZ.

JACK SMITH, BRITISH NMO, 2017

102. Consider a cyclic quadrilateral $ABCD$. The diagonals AC and BD meet at P and the rays AD and BC meet at Q. The internal angle bisector of angle $\sphericalangle BQA$ meets AC at R and the internal angle bisector of angle $\sphericalangle APD$ meets AD at S. Prove that RS is parallel to CD.

DAVID MONK, BRITISH NMO, 2017

103. Points M and N are chosen on the sides BC and CD of a square $ABCD$, respectively, so that $\sphericalangle MAN = 45°$. Circle w with diameter MN intersects segments AM and AN at points P and Q, respectively. Prove that points B, P and Q lie on a single line.

UKRAINIAN NMO, 2017

104. In a trapezoid $ABCD$ with bases AD and BC, the angle bisector of $\sphericalangle DAB$ intersects the angle bisectors of $\sphericalangle ABC$ and $\sphericalangle CDA$ at points P and S, respectively and the angle bisector of $\sphericalangle BCD$ intersects the angle bisectors of $\sphericalangle ABC$ and $\sphericalangle CDA$ at points Q and R, respectively. Prove that if $PS \parallel RQ$, then $AB = CD$.

UKRAINIAN NMO, 2017

105. We consider the right triangle ABC with the right angle in A. Let D be the contact point of altitude from A on BC and E the bisector's intersection of the angle $\sphericalangle ADC$ with AC. If $M \in (AE)$, $N \in (DC)$ and $\{F\} = MN \cap AD$ such that $\frac{AE}{DC} = \frac{ME}{NC} = k > 1$, find in function of k the rapport $\frac{BD}{AF}$.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

106. Let A, B, C, D be the measures in grades of the angles of a convexe quadrilateral. Prove that:

$$\frac{A^2+B^2+C^2}{360°-D}+\frac{B^2+C^2+D^2}{360°-A}+\frac{C^2+D^2+A^2}{360°-B}+\frac{D^2+A^2+B^2}{360°-C} \geq 360°.$$

DANIEL SITARU, RMM, ROMANIA

107. Let AA' be median in triangle ABC, $A' \in (BC)$. Let be $M \in (BA')$, $N \in (A'C)$. The parallel through M to AA' intersects AB and AC in S, respectively T. The parallel through N to AA' intersects AC and AB in S', respectively T'. Prove that:

$$MS + MT = NS' + NT'.$$

DANIEL SITARU, RMM, ROMANIA

108. a) Find $a, b, c, d, e, f \in \mathbb{N}^*$ such that $2014 = a^2 + b^2 - c^2$; $2015 = d^2 + e^2 - f^2$.

b) Prove that if $n \in \mathbb{N}$, $n \geq 7$, it exists $a, b, c \in \mathbb{N}^*$ such that $n = a^2 + b^2 - c^2$.

DANIEL SITARU, RMM, ROMANIA

109. Find $a, b, c, d, e \in \mathbb{Z}$, such that:

$$|a-b| = |b-c| = |d-e| = |e-a|;\ abcde = 1.$$

DANIEL SITARU, RMM, ROMANIA

110. Solve the equation:

$$x^3 + 2013^3 + 2014^3 = 4054182x.$$

DANIEL SITARU, RMM, ROMANIA

111. Prove that the number $N = \overline{\underbrace{111...1}_{\text{for 2010 times}}}$ is composed.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

112. Prove that the number $N = \overline{\underbrace{999...9}_{\text{for 2011 times}}} + \overline{1\underbrace{999...9}_{\text{for 2011 times}}\underbrace{000...0}_{\text{for 1005 times}}}$ is composed.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

113. We consider an ABC triangle with the lengths sides $AB = 16$, $BC = 34$, $CA = 30$ and the points M, respectively N on the side BC such that $BM = 4$, $MN = 12$. Compute the measure of the angle $\sphericalangle MAN$.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

114. We denote "*Super-Heron*" triangle (do not confuse with "Super-Hero") a triangle that has the lengths sides consecutive natural numbers and the area also a natural number.
Recently[1] (2007), was proved the existence of an infinite number of such triangles.
Confirm or infirm the existence of "Super-Heron" inscriptible quadrilaterals.
(*Indication.* Use the area formula of the inscriptible quadrilateral given by the Indian mathematician *Brahamgupta* in VII$^{\text{th}}$ century – A.H.: $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where a, b, c, d are the quadrilateral lengths sides and s is its semiperimeter).

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

115. In triangle ABC, $m(\sphericalangle A) = 90°$, $m(\sphericalangle B) = 15°$, $AD \perp BC$, $D \in (BC)$, $DE \perp AB$, $E \in (AB)$, $EF \perp AD$, $F \in (AD)$, $EF = 4$ cm. Solve triangle ABC.

DANIEL SITARU, RMM, ROMANIA

116. Prove that an inscriptible quadrilateral having the sides $AB = 3$, $BC = 4$, $CD = 5$, $AD = 2x$, $x \in \mathbb{N}^*$ cannot have the aria equal to $\sqrt{2013}$ for no natural value of x.

DANIEL SITARU, RMM, ROMANIA

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[1] http://www.math.twsu.edu/~richardson/heronian/heronian.html

117. Find the biggest $n \in \mathbb{N}^*$ such that:

$$\frac{1}{\sqrt{2+\sqrt{3}}}+\frac{1}{\sqrt{4+\sqrt{15}}}+\ldots+\frac{1}{\sqrt{2n+\sqrt{4n^2-1}}}<2013\sqrt{2}.$$

DANIEL SITARU, RMM, ROMANIA

118. We consider a right-angled triangle with the perimeter P and the area A natural numbers. Prove that the hypotenuse is a natural number if and only if P is natural even number and $\frac{P}{2}$ divides $A \Leftrightarrow P|2A$.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

119. Which conditions must meet the triangle's angle so this can be divided in two isosceles triangles?
Find the angles of an isosceles triangle which can be divided in two isosceles triangles.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

120. Find the biggest triangle with the dimensions natural numbers, whose sum is even number for which the area is also a natural number and it is equal with its perimeter.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

121. Prove that if a, b, c are the lengths of the sides of a triangle and:

$$(a-b)^{2015}(a-c)^{2015}+(b-a)^{2015}(c-a)^{2015}+(c-a)^{2015}(c-b)^{2015}=0,$$

then the triangle is equilateral.

DANIEL SITARU, RMM, ROMANIA

122. Prove that if $\frac{7^n-7}{n} \in \mathbb{N}$, $n \in \mathbb{N}^*$, then:

$$\frac{7^{7^n}-7^7}{7^n-1} \in \mathbb{N}.$$

DANIEL SITARU, RMM, ROMANIA

123. We have a trapeze with the bases AB = 17 cm, CD = 8 cm and $MN \parallel AB$, $M \in (AD)$, $N \in (BC)$, $\frac{BN}{CN}=3.$ Compute MN.

DANIEL SITARU, RMM, ROMANIA

124. Solve in prime natural numbers the equation $p^r = p+2q$.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

125. Prove that for any even natural number, perfect square, written as a sum of two natural numbers, the maximum of the product of the two terms of the sum is also a natural even number, perfect square.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

126. Which of the numbers: $2010^{2010}, 2010^{2011}, 2011^{2010}$ and 2011^{2011} can be written as a sum of two perfect squares?

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

127. A cross-country race was scheduled to start at 5:00 PM and x minutes. With how many minutes was the race won, if the finish line was passed in the same day, also at 5:00 PM, but at y minutes, knowing that both at the start and at the finish the angle between minute hand and the hour hand was 66°?

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

128. Find the maximum value of a product of some nonzero natural numbers whose sum is equal to 2013?

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

129. Let be the numbers $2^{2010} = \overline{x_1 x_2 \ldots x_m}$ and $5^{2010} = \overline{y_1 y_2 \ldots y_n}$ written in base ten. How many digits has the number $N = \overline{x_1 x_2 \ldots x_m y_1 y_2 \ldots y_n}$?

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

130. Find $x, n \in \mathbb{N}^*$ such that:

$$\frac{x}{0,(x)} + \frac{x^2}{0,0(x)} + \frac{x^3}{0,00(x)} + \ldots + \frac{x^n}{0,\underbrace{000\ldots00}_{\text{for } n-1 \text{ times}}(x)} = 3789 .$$

DANIEL SITARU, RMM, ROMANIA

131. Let be $a, b, c, d \in (0, \infty)$, $a + b + c + d = 2$. Prove that:

$$\sqrt{a(b+c+d)} + \sqrt{b(a+c+d)} + \sqrt{c(a+b+d)} + \sqrt{d(a+b+c)} \le 4 .$$

Generalization: Let be $x_1, x_2, \ldots, x_n \in (0, \infty)$, $x_1 + x_2 + \ldots + x_n = 2$, $n \in \mathbb{N}$.

In these conditions:

$$\sqrt{x_1(x_2 + x_3 + \ldots + x_n)} + \sqrt{x_2(x_1 + x_3 + \ldots + x_n)} + \ldots + \sqrt{x_n(x_1 + x_2 + \ldots + x_{n-1})} \le n .$$

DANIEL SITARU, RMM, ROMANIA

132. Prove that the arithmetic means of the firsts n decimals of the number $\sqrt{3} - 1$ is contained between $4\frac{2}{5}$ and $4\frac{3}{5}$, then the arithmetic means of the firsts n decimals of the number $2 - \sqrt{3}$ has the same property.

DANIEL SITARU, RMM, ROMANIA

133. Prove that if $a, b, c \in \mathbb{R}$ and $1 - 4a + 3b \geq 0$, $a - 4b + 3c \geq 0$, $b - 4x + 3a \geq 0$, $c - 4b + 3a \geq 0$, then $a = b = c = 1$.

DANIEL SITARU, RMM, ROMANIA

134. Solve the equation:

$$\frac{x-2014}{2}+\frac{x-2010}{3}+\frac{x-2004}{4}+\ldots+\frac{x-1985}{11}=55.$$

DAN NĂNUȚI, RMM, ROMANIA

135. Solve in $\mathbb{R}$ the system: $\begin{cases} x = yzt \\ y = xzt \\ z = xyt \\ t = xyz \end{cases}$.

CLAUDIA NĂNUȚI, RMM, ROMANIA

136. Let be $C = 9\overline{\underbrace{11\ldots1}_{2015\text{ times}}\underbrace{aa\ldots a}_{2015\text{ times}}\underbrace{bbb\ldots b}_{2015\text{ times}}}+8$. Find $a, b \in \mathbb{N}$ such that C would be a perfect cube.

DANIEL SITARU, RMM, ROMANIA

137. Prove that in an ABC triangle we have $a+\alpha h_a = b+\alpha h_b = c+\alpha h_c,\ \alpha \in (0, \infty)$, then the triangle is equilateral.

DANIEL SITARU, RMM, ROMANIA

138. Let be $A = 1234567891011\ldots20142015$. Prove that $\sqrt{A}$ is an irational number.

DANIEL SITARU, RMM, ROMANIA

139. Prove that if $a, b, c \in (0, \infty)$, then:

$$\left(a^2+a+1\right)\left(b^2+b+1\right)\left(c^2+c+1\right) \geq 27abc.$$

DANIEL SITARU, RMM, ROMANIA

140. Prove that if $a, b, c, d, x, y, z, t \in (0, \infty)$,

$$\sqrt{ax}+\sqrt{by}+\sqrt{cz}+\sqrt{dt}=\sqrt{(a+b+c+d)(x+y+z+t)},$$

then a, b, c, d are direct proportional with x, y, z, t.

DANIEL SITARU, RMM, ROMANIA

141. Let be $a, b, c, d \in (0, \infty)$. Prove that:

$$\frac{a}{a+2b}+\frac{b}{2a+b}+\frac{c}{c+2d}+\frac{d}{2c+d} \geq \frac{4}{3}.$$

DANIEL SITARU, RMM, ROMANIA

142. Let be $A=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{100}}$. Prove that $18-A$ is not a natural number.

DANIEL SITARU, RMM, ROMANIA

143. Let be $a, b, c \in \mathbb{N}^* \setminus \{1\}$. Prove that:

$$\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{b^2}\right)\left(1-\frac{1}{c^2}\right)>\frac{1}{8}.$$

DANIEL SITARU, RMM, ROMANIA

144. Prove that if $a, b, c, d, e \in \mathbb{R}$, $a+2b+3c+4d+5e \geq 55$, then:

$$a^2+b^2+c^2+d^2+e^2 \geq 55.$$

DANIEL SITARU, RMM, ROMANIA

145. Let be $a, b, c \in \mathbb{R}$, $a^2+b^2+c^2=2015$. Prove that:

$$\sqrt{2015-a^2}+\sqrt{2015-b^2}+\sqrt{2015-c^2} \geq \sqrt{2}(a+b+c).$$

DANIEL SITARU, RMM, ROMANIA

146. Let be $a, b, c, d \in \mathbb{R}$, $a^2+b^2+c^2+d^2=2016$. Prove that:

$$\sqrt{2016-a^2}+\sqrt{2016-b^2}+\sqrt{2016-c^2}+\sqrt{2016-d^2} \geq \sqrt{3}(a+b+c+d).$$

DANIEL SITARU, RMM, ROMANIA

147. Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n \in \mathbb{N}^*$ and $4^n \cdot x_1x_2\ldots x_n = 1$. Prove that:

$$4^n\left(x_1^2+x_2\right)\left(x_2^2+x_3\right)\cdot\ldots\cdot\left(x_{n-1}^2+x_n\right)\left(x_n^2+x_1\right) \geq 1.$$

DANIEL SITARU, RMM, ROMANIA

148. Let be $x_1, x_2, \ldots, x_n$ positive numbers, $x_1+x_2+\ldots+x_n=1$, $n \in \mathbb{N}^*$. Prove that for any $p \in \mathbb{R}$ we have:

$$2\left(\sqrt{p^2x_1+1}+\sqrt{p^2x_2+1}+\ldots+\sqrt{p^2x_n+1}\right) \leq p^2+2n.$$

DANIEL SITARU, RMM, ROMANIA

149. Graphically represent the sets:

$$A=\{(x, y) \mid x, y \in [-1, 1];\ x|x| = y|y|\};$$

$$B=\{(x, y) \mid x, y \in [-1, 1];\ x|x| = -y|y|\}.$$

DANIEL SITARU, RMM, ROMANIA

150. Let be $x_1, x_2, \ldots, x_{2n+1} \in \mathbb{Z}$, $n \in \mathbb{N}^*$, $x_1x_2 \cdot \ldots \cdot x_{2n+1} = 1$. Find $x_1, x_2, \ldots, x_{2n+1}$ such that:

$$|x_1 - x_2| = |x_2 - x_3| = \ldots = |x_{2n} - x_{2n+1}| = |x_{2n+1} - x_1|.$$

DANIEL SITARU, RMM, ROMANIA

151. Prove that:

$$\left(1+\frac{\pi}{e}\right)^9 + \left(1+\frac{e}{\pi}\right)^9 > 1024.$$

DANIEL SITARU, RMM, ROMANIA

152. Let be $x, y \in \mathbb{Z}$, $|x| \le 11$, $|y| \le 100$. Prove that:

$$\left|x\sqrt{5} + y\sqrt{7}\right| > \frac{1}{333}.$$

DANIEL SITARU, RMM, ROMANIA

153. Let be $x, y, z \in [0, \infty)$. Prove that:

$$\frac{x}{(1+x)^2} + \frac{y}{(1+x+y)^2} + \frac{z}{(1+x+y+z)^2} \le \frac{x+y+z}{1+x+y+z}.$$

DANIEL SITARU, RMM, ROMANIA

154. Prove that: $(\forall)\ x, y, z \in (0, \infty)$,

$$x^3 + y^3 + z^3 + xyz(xy + yz + xz) \ge 8\left(\frac{y^2z^2\sqrt{x}}{(z+y)^2} + \frac{x^2y^2\sqrt{z}}{(x+y)^2} + \frac{z^2x^2\sqrt{y}}{(z+x)^2}\right).$$

DANIEL SITARU, RMM, ROMANIA

155. Prove that if x, y and z are in $[-5, 3]$, then:

$$\sqrt{3x-5y-xy+15} + \sqrt{3y-5z-yz+15} + \sqrt{3z-5x-xz+15} \le 12.$$

When does equality occur?

HUNGARIAN NMO, 2017

156. Let $a, b, c \ge -1$ be real numbers with $a^3 + b^3 + c^3 = 1$. Prove that:

$$a + b + c + a^2 + b^2 + c^2 \le 4.$$

When does equality holds?

SWEDEN NMO, 2017

157. If in ΔABC, h_1, h_2, h_3 are altitudes of $\Delta C_AC_BC_C$, where AC_A, BC_B, CC_C – Gergonne's cevians, then:

$$\left(\frac{h_1}{h_a}\right)^2 + \left(\frac{h_2}{h_b}\right)^2 + \left(\frac{h_3}{h_c}\right)^2 = 1 - \frac{r}{2R}.$$

MEHMET SAHIN, ANKARA, TURKEY

158. Let O be the circumcenter of triangle ADC. A tangent t to the circumcircle of triangle DOC meets the sides AD and AC at points D and E, respectively (D, $E \not\equiv A$). Point A' is the reflection of A in line t. Prove that the circumcircles of triangles $A'DE$ and ADC are tangent to each other.

DUŠAN DJUKIĆ, SERBIAN NMO, 2016

159. In a triangle ADC ($AD \neq AC$), the incircle with center I is tangent to side BC at point D. Let M be the midpoint of side BC. Prove that the perpendiculars from points M and D to lines AI and MI, respectively, meet on the altitude in ΔABC from A or its extension.

DUŠAN DJUKIĆ, SERBIAN NMO, 2016

160. Let h_1, h_2, h_3, m_1, m_2, m_3 be the altitudes, respectively the medians of intouch triangle in ΔABC. Prove that:

$$\frac{\frac{1}{h_1^2}+\frac{1}{h_2^2}+\frac{1}{h_3^2}}{\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}}=\frac{4R^2}{r^2}\cdot\frac{m_1^2+m_2^2+m_3^2}{m_a^2+m_b^2+m_c^2}.$$

MEHMET SAHIN, RMM, TURKEY

161. If $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in (0, \infty)$, $n \in \mathbb{N}^*$, then:

$$\left(\sum_{i=1}^{n}\frac{x_i^2+y_i^2}{x_iy_i}\right)\left(\sum_{i=1}^{n}\frac{x_iy_i}{x_i^2+y_i^2}\right)\le\left(\sum_{i=1}^{n}\frac{x_i}{y_i}\right)\left(\sum_{i=1}^{n}\frac{y_i}{x_i}\right).$$

DANIEL SITARU, RMM, ROMANIA

162. If $a, b, c \in (0, \infty)$, then:

$$\left(a^4+b^4+c^4\right)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right)\ge 2\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right).$$

DANIEL SITARU, RMM, ROMANIA

163. If $a_k \in (0, \infty)$, where $k = 1, 2, 3, \ldots, n$ and $a_1 + a_2 + \ldots + a_n = 1$, then:

$$\sum_{k=1}^{n}\frac{1}{a_k}\ge(n+1)\left(\sum_{k=1}^{n}\frac{1}{a_k+1}\right).$$

MARIN CHIRCIU, RMM, ROMANIA

164. If $a_i, b_i \in (0, \infty)$, $i\in\overline{1,n}$, $n \in \mathbb{N}^*$, then:

$$\frac{(2n)^n}{\prod_{i=1}^{n}(a_i+b_i)}\le\frac{1}{2}\left[\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)^n+\left(\sum_{i=1}^{n}\frac{1}{b_i}\right)^n\right].$$

DANIEL SITARU, RMM, ROMANIA

165. If $a_i > 0$, $i \in \overline{1, n}$, $a_1 + a_2 + \ldots + a_n = 1$, then:

$$\frac{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}{\frac{1}{1+a_1}+\frac{1}{1+a_2}+\ldots+\frac{1}{1+a_n}} \geq n+1.$$

REGRAGUI EL KHAMMAL, RMM, MOROCCO

166. Let *ABC* be an acute, non-isosceles triangle with (*O*) its circumcircle. Denote *H* at the orthocenter and *BE*, *CF* as the altitudes of triangle *ABC.* Suppose that *AH* intersects (*O*) at *D* differs from *A*.

1. Let *I* be the midpoint of *AH*, *EI* meets *BD* at *M* and *FI* meets *CD* at *N*. Prove that *MN* is perpendicular to *OH*.

2. The lines *DE*, *DF* intersect (*O*) at *P*, *Q*, respectively (*P* and *Q* differ from *D*). The circle (*AEF*) intersects (*O*) and *AO* at *R*, *S*, respectively (*R* and *S* differ from *A*). Prove that *BP*, *CQ*, *RS* are concurrent.

VIETNAMESE NMO, 2017

167. If $a_1 = 1$, $a_2 = 2$ and for any natural number $n \geq 2$, we have:

$$b_{n+1} = a_n,\ a_{n+1} = b_n + c_n,\ c_{n+1} = 2a_n,\ d_n = 2a_n + 2b_n + c_n,$$

then find d_{2015} and d_{2016}.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

168. Let *ABCD* be a rectangular trapeze $AB \parallel CD$, $AB < CD$, $AB = a$, $AD = b$, $CD = c$ and *P* a variable point on [*AD*]. Find the minimum sum $BP + PC$.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

169. Find the lengths of the sides of a right-angled triangle, knowing that there are natural numbers and the lengths of the hypotenuse is equal with the difference between the lengths of the product of the cathetus and their sum.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

170. Prove that if the area of a square can be written as a sum of *n* consecutive terms (natural numbers), then *n* is less than the lengths of the square's diagonal.

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA

171. The inradii of *ABC* triangle is $r = a$ cm, where $a > 0$. The points of tangents of the circle with the triangle's sides are $M \in AB$, $N \in BC$ and $P \in CA$. Knowing that $AP = a$ cm and $PC = 3a$ cm, find the area of *ABC* triangle.

MARIN CHIRCIU, ROMANIA