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PROBLEMS

351. Let a, b and c be the lengths of the sides of a triangle with inradius r. Prove that:

$$\frac{a^3}{(a-b-c)^2}+\frac{b^3}{(a-b+c)^2}+\frac{c^3}{(a+b-c)^2}\geq 6\sqrt{3}r.$$

George Apostolopoulos

352. Prove that in any ABC triangle the following relationship is true:

$$a^{2+\sqrt{3}}+b^{2+\sqrt{3}}+c^{2+\sqrt{3}}\geq 3\cdot\left(2r\sqrt{3}\right)^{2+\sqrt{3}}.$$

Daniel Sitaru

353. Let a, b, c be the lengths of the sides of a triangle ABC with circumradius R. Prove that:

$$\frac{1}{(a+b)(b+c)}+\frac{1}{(b+c)(c+a)}+\frac{1}{(c+a)(a+b)}\geq\frac{1}{4R^2}.$$

George Apostolopoulos

354. Prove that in any ABC triangle the following relationship is true:

$$a^6+b^6+c^6\geq 81\cdot(2r)^6.$$

Daniel Sitaru

355. Let ABC be a triangle with $\sphericalangle C=2\sphericalangle B$. On side BC consider a point D such that $\sphericalangle BAD=\frac{1}{4}\sphericalangle C$. Prove that:

a. $BD=\frac{AB\cdot AC}{AB+AC}$; b. $\frac{\cos C}{\cos B}\geq\frac{\cot C}{\cot B}$.

George Apostolopoulos

356. Prove that in any $\Delta\ ABC$ the following relationship is true:

$$\frac{1}{\sqrt{\sin A}}+\frac{1}{\sqrt{\sin B}}+\frac{1}{\sqrt{\sin C}}\leq\frac{a^2+b^2+c^2+3}{2\sqrt{2}F}.$$

Daniel Sitaru

357. Let R, r, r_a, r_b, r_c represent the circumradius, inradius and exradii, respectively, of $\Delta\ ABC$. Prove that:

a. $\left(1+\cot\frac{A}{2}\right)\left(1+\cot\frac{B}{2}\right)\left(1+\cot\frac{C}{2}\right)\geq\left(1+2\sqrt{3}\frac{r}{R}\right)^3$;

b. $\left(1+\frac{1}{r_a}\right)\left(1+\frac{1}{r_b}\right)\left(1+\frac{1}{r_c}\right)\geq\left(1+\frac{2}{3R}\right)^3$.

George Apostolopoulos

358. Prove that in any $\Delta\ ABC$ the following relationship is true:

$$\sum\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right)^3 \geq \sqrt[3]{3a}+\sqrt[3]{3b}+\sqrt[3]{3c}-2.$$

Daniel Sitaru

359. For an acute triangle ABC and a positive integer n, prove that:

$$\left(\sum(\sin A\sin B\sin C)^{\frac{1}{n}}\right)^n \leq \frac{3^{n+1}}{8},$$

where the sum is over all cyclic permutations of (A, B, C).

George Apostolopoulos

360. Prove that if $m, n \in \mathbb{N}^*$ then in any triangle the following relationship holds:

$$2R\sum w_a^{m+1}\left(w_a^{2n}-w_a^n+1\right) \geq \frac{(ab+bc+ca)^{m+n+1}}{(6R)^{m+n}}.$$

Daniel Sitaru

361. Let ABC be a triangle. Prove that:

$$\left(1+\sec\frac{A}{2}\right)^{\csc^{-3}\frac{A}{2}}\cdot\left(1+\sec\frac{B}{2}\right)^{\csc^{-3}\frac{B}{2}}\cdot\left(1+\sec\frac{C}{2}\right)^{\csc^{-3}\frac{C}{2}} \geq \left(5+\frac{26\sqrt{3}}{9}\right)^{\frac{1}{8}}.$$

George Apostolopoulos

362. Prove that in any $\Delta\ ABC$ the following relationship holds:

$$\sum m_a h_a w_a \geq \sin A\sin B\sin C\left(\frac{b^2c^2}{a}+\frac{c^2a^2}{b}+\frac{a^2b^2}{c}\right).$$

Daniel Sitaru

363. Let ABC be a triangle. Prove that:

$$\left(1+\sec\frac{A}{2}\right)^{8\sin^3\frac{A}{2}}\cdot\left(1+\sec\frac{B}{2}\right)^{8\sin^3\frac{B}{2}}2\cdot\left(1+\sec\frac{C}{2}\right)^{8\sin^3\frac{C}{2}} \geq 5+\frac{26}{9}\sqrt{3}.$$

George Apostolopoulos

364. Prove that in any ΔABC the following relationship holds:

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} > \frac{1}{2R}(m_a+m_b+m_c).$$

Daniel Sitaru

365. Let P be an arbitrary point inside a triangle ABC. Let a, b and c be the distances from P to the side BC, AC and AB, respectively. Prove that:

$$\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^4}{\sin^4 A+\sin^4 B+\sin^4 C} \leq 12R^2,$$

where R denotes the circumradius of ABC.

When does the equality occur?

George Apostolopoulos

366. Prove that in ABC triangle the following relationship holds:

$$\sum m_a\left(|\sin B|\sqrt{1+\cos^2 B}+|\cos B|\sqrt{1+\sin^2 B}\right)<2s\sqrt{3}.$$

Daniel Sitaru

367. Let ABC be a given triangle, and let M, N be the interior points on the side BC such that $BM = CN$. Prove that:

$$\frac{1}{AM}+\frac{1}{AN}>\frac{4}{AB+AC}.$$

George Apostolopoulos

368. Prove that in any ABC triangle the following relationship holds:

$$\sqrt{m_a}+\sqrt{m_b}+\sqrt{m_c}\geq\sqrt{m_a+m_b+m_c+6\sqrt[3]{h_a h_b h_c}}.$$

Daniel Sitaru

369. Let ABC be a given triangle, and let M, N be the interior points on the side BC such that $BM = NC$. Prove that:

$$\frac{AC^2}{AM}+\frac{AB^2}{AN}>BC.$$

George Apostolopoulos

370. Prove that in any ΔABC the following relationship holds:

$$\sum\sqrt{a+b-c}\geq\sqrt{2p+12\sqrt[3]{rF}}.$$

Daniel Sitaru

371. In an acute triangle ABC heights AA_1, BB_1, and CC_1 are drawn. Let H be the intersection point of heights. Prove that:

$$\frac{AH}{HA_1}+\frac{BH}{HB_1}+\frac{CH}{HC_1}\geq\sec A+\sec B+\sec C.$$

George Apostolopoulos

372. Prove that in any ABC triangle the following relationship holds:

$$\frac{b^4c^7}{a^{12}}+\frac{c^4a^7}{b^{12}}+\frac{a^4b^7}{c^{12}}\geq\frac{\sqrt{3}}{R}.$$

Daniel Sitaru

373. Let ABC be an acute triangle with inradius r and circumradius R. Prove that:

$$\frac{(\sec A)^{\frac{1}{\sec A}}+(\sec B)^{\frac{1}{\sec B}}+(\sec C)^{\frac{1}{\sec C}}}{\sec A+\sec B+\sec C}<\frac{5R-r}{12r}.$$

George Apostolopoulos

374. Prove that in any ABC triangle the following relationship holds:

$$(m_a^2+m_a m_b+m_b^2)(m_b^2+m_b m_c+m_c^2)(m_c^2+m_c m_a+m_a^2)\geq\left(\frac{9r_a r_b r_c}{r_a+r_b+r_c}\right)^3.$$

Daniel Sitaru

375. Let ABC be an acute triangle with inradius r and circumradius R. Prove that:

$$\sec A + \sec B + \sec C \geq 12\frac{r}{R}.$$

George Apostolopoulos

376. Prove that in ABC and $A'B'C'$ triangles $a + a' = b + b' = c + c'$ having the sides a, b, c, respectively a', b', c', then:

$$s(a'b' + b'c' + c'a') \leq 6R'F'\left(\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\right),$$

where $s = \dfrac{a+b+c}{2}$; $F' = \text{Area}\,[A'B'C']$.

Daniel Sitaru

377. Prove that in all acute triangles ABC holds

$$\tan^2 A \cdot \tan^2 B \cdot \tan^2 C \cdot \cos(A - B) \cdot \cos(B - C) \cdot \cos(C - A) \geq 27.$$

George Apostolopoulos

378. Prove that in ABC triangle the following relationship holds:

$$\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

Daniel Sitaru

379. Let w_a, w_b, w_c be the lengths of the internal bisectors of the angles opposite of the sides a, b, c of a triangle ABC. Prove that:

$$\left(\frac{w_a}{bc}\right)^n + \left(\frac{w_b}{ca}\right)^n + \left(\frac{w_c}{ab}\right)^n \leq \frac{3}{(4r)^n}$$

for each positive integer n.

George Apostolopoulos

380. Prove that in any ABC triangle the following relationship holds:

$$\frac{3}{2} \leq \sum \frac{a\sqrt{(b-c)^2 + 4r^2}}{b\sqrt{(a-c)^2 + 4r^2} + c\sqrt{(b-a)^2 + 4r^2}} < 2.$$

Daniel Sitaru

381. Let a, b, c be the lengths of sides, and let m_a, m_b, m_c be the lengths of medians of a triangle ABC with inradius r and circumradius R. Prove that:

$$24\frac{r^2}{R} \leq \frac{a^2}{\sqrt{m_b m_c}} + \frac{b^2}{\sqrt{m_c m_a}} + \frac{c^2}{\sqrt{m_a m_b}} \leq \frac{3R\sqrt{2R(R-r)}}{r}.$$

George Apostolopoulos

382. Prove that in ABC triangle the following relationship holds:

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \leq \frac{1}{4F}\sum\left(\frac{\sqrt{a}(a^2 + bc)}{\sqrt{b} + \sqrt{c}} + a^2\right).$$

Daniel Sitaru

383. Let m_a, m_b, m_c be the lengths of the medians of a triangle with circumradius R. Prove that:

$$\left(1+\frac{1}{m_a}\right)\left(1+\frac{1}{m_b}\right)\left(1+\frac{1}{m_c}\right) \geq \left(1+\frac{2}{3R}\right)^3.$$

George Apostolopoulos

384. In $ABCD$ inscriptible quadrilateral, we denote $AB = a; BC = b; CD = c;$ $DA = d;\ F = Area\,[ABCD]$. Prove that:

$$\sin A + \sin B + \sin C + \sin D \leq \frac{4F}{\sqrt{abcd}}.$$

Daniel Sitaru

385. Let ABC be an acute triangle. Prove that:

$$\sqrt{\cos A \cdot \sin B \cdot \sin C} + \sqrt{\sin A \cdot \cos B \cdot \sin C} + \sqrt{\sin A \cdot \sin B \cdot \cos C} \leq \frac{3}{2}\sqrt{\frac{3}{2}}.$$

George Apostolopoulos

386. In $ABCD$ inscriptible quadrilateral, we denote $AB = a; BC = b; CD = c;$ $DA = d; s = \dfrac{a+b+c+d}{2}$. Prove that:

$$\sin A \sin B \leq \left(1-\frac{s}{a}\right)\left(1-\frac{s}{b}\right)\left(1-\frac{s}{c}\right)\left(1-\frac{s}{d}\right).$$

Daniel Sitaru

387. Let ABC denote a triangle, I its incenter, R its circumradius, r its inradius, and x, y and z the inradii of triangle IBC, ICA and IAB respectively. Prove that:

$$\frac{\sin A}{x} + \frac{\sin B}{y} + \frac{\sin C}{z} \leq \frac{4+3\sqrt{3}}{2r} + \frac{2}{R}.$$

George Apostolopoulos

388. Prove that in any $ABCD$ inscriptible quadrilateral $AB = a;\ BC = b; CD = c;$ $AD = d; F = Area\,[ABCD]$ the following relationship holds:

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{D}{2} \leq \frac{F^2}{4abcd}.$$

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389. Let a, b, c be the side lengths of a triangle ABC with circumradius R and inradius r. Prove that:

$$\frac{1}{4R^2} \leq \frac{1}{(a+b)(b+c)} + \frac{1}{(b+c)(c+a)} + \frac{1}{(c+a)(a+b)} \leq \frac{1}{16r^2}.$$

George Apostolopoulos

390. Prove that in any $ABCD$ inscriptible quadrilateral $AB = a;\ BC = b; CD = c;$ $AD = d; F = Area\,[ABCD]$ the following relationship holds:

$$a^2 - b^2 - c^2 + d^2 + 4F \leq 2\sqrt{2}(ad + bc)$$

Daniel Sitaru

391. Let a, b and c be the side lengths of a triangle ABC with circumradius R and inradius r. Prove that:

a. $\frac{1}{a}\sqrt{\frac{a}{b+c-a}}+\frac{1}{b}\sqrt{\frac{b}{c+a-b}}+\frac{1}{c}\sqrt{\frac{c}{a+b-c}}\geq\frac{\sqrt{3}}{R}$;

b. $\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin^2 C}\leq\left(\frac{R}{r}\right)^2$.

George Apostolopoulos

392. Prove that in any ABC acute-angled triangle the following relationship holds:

$$\sum(\tan A+\tan B)(\tan B+\tan C)>6\tan A\tan B\tan C.$$

Daniel Sitaru

393. If $0<a\leq b$ then:

$$\left(1+\frac{a+3b}{4}\right)^{3a+b}\leq\left(1+\frac{3a+b}{4}\right)^{a+3b}.$$

Daniel Sitaru

394. Let be $P\in Int\ (\Delta ABC)$; $PA=x$; $PB=y$; $PC=z$. Prove that:

$$(ax+by-cz)(ax-by+cz)(by+cz-ax)\leq abcxyz$$

where $AB=c$; $BC=a$; $CA=b$.

Daniel Sitaru

395. Let a, b and c be the side lengths of a triangle ABC with inradius r. Prove that:

$$\frac{1}{\left(a+b-\sqrt{ab}\right)^2}+\frac{1}{\left(b+c-\sqrt{bc}\right)^2}+\frac{1}{\left(c+a-\sqrt{ca}\right)^2}\leq\frac{1}{4r^2}.$$

George Apostolopoulos

396. Prove that in any ABC triangle the following relationship holds:

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}>\frac{1}{2R}(m_a+m_b+m_c).$$

Daniel Sitaru

397. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r, circumradius R, and let m_a, m_b, m_c be the lengths of the medians of triangle ABC. Prove that:

$$9\sqrt{3}r\cdot\sqrt[3]{\frac{r}{4R}}\leq\frac{m_a^2}{a}+\frac{m_b^2}{b}+\frac{m_c^2}{c}\leq\frac{9R}{8r}\sqrt{6R\cdot(R-r)}.$$

George Apostolopoulos

398. Prove that in ABC triangle the following relationship holds:

$$\frac{1}{b+c}h_a\cos A+\frac{1}{c+a}h_b\cos B+\frac{1}{a+b}h_c\cos C<\sum\frac{bc}{b^2+c^2}.$$

Daniel Sitaru

399. Let a, b, c be the lengths of sides of a triangle ABC with semiperimeter s and inradius r and let x, y, z be the distances from the incentre of ΔABC to the vertices A, B, C respectively. Prove that:

$$\frac{9\sqrt{2\cdot r}}{2s} \le \frac{\sqrt{x}}{a} + \frac{\sqrt{y}}{b} + \frac{\sqrt{z}}{c} \le \frac{1}{r}\sqrt{\frac{s\cdot\sqrt{3}}{6}}.$$

George Apostolopoulos

400. Prove that in ABC triangle the following relationship holds:

$$\frac{h_b^5 h_c^4}{h_a^{10}} + \frac{h_c^5 h_a^4}{h_b^{10}} + \frac{h_a^5 h_b^4}{h_c^{10}} \ge 3\sqrt[3]{\frac{R}{2S^2}}.$$

Daniel Sitaru

401. Let a, b, c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that:

$$(b^4 + c^4)\sin^2 A + (c^4 + a^4)\sin^2 B + (a^4 + b^4)\sin^2 C \le \frac{81}{4}(3R^4 - 16r^4).$$

George Apostolopoulos

402. Prove that in any ABC triangle the following relationship holds:

$$\frac{1}{2}(ab + bc + ca) \le 2sR + 2F\left(\frac{\cos B}{\sin A} + \frac{\cos C}{\sin B} + \frac{\cos A}{\sin C}\right).$$

Daniel Sitaru

403. A triangle with side lengths a, b, c has perimeter equal to 3. Prove that:

a. $\dfrac{(9-4a)(9-4b)(9-4c)}{(a+b)^2(b+c)^2(c+a)^2} < \dfrac{64}{27}$;

b. $a^3 + b^3 + c^3 + a^4 + b^4 + c^4 \ge 2(a^2b^2 + b^2c^2 + c^2a^2)$.

George Apostolopoulos

404. Prove that in any ABC triangle the following relationship holds:

$$\prod \frac{(a+b-c)^3 + (b+c-a)^3}{(a+b-c)^2 + (a+b-c)(b+c-a) + (b+c-a)^2} \ge \frac{8abc}{27}.$$

Daniel Sitaru

405. Let ABC be a triangle with inradius r and circumradius R. Prove that:

a. $\dfrac{8\sqrt{3}}{3} \le \dfrac{1}{\cos^3\frac{A}{2}} + \dfrac{1}{\cos^3\frac{B}{2}} + \dfrac{1}{\cos^3\frac{C}{2}} \le \dfrac{2\sqrt{3}}{3}\left(\dfrac{R}{r}\right)^2$;

b. $9\left(\dfrac{r}{R}\right)^2 \le \cos^2\dfrac{A}{2} + \cos^2\dfrac{B}{2} + \cos^2\dfrac{C}{2} \le \dfrac{9}{4}$.

George Apostolopoulos

406. Prove that if $x, y, z \in (0, \infty)$ then:

$$\frac{x^6}{(2y+3z)^6\sin^4\frac{\pi}{18}} + \frac{y^6}{(2z+3x)^6\cos^4\frac{\pi}{9}} + \frac{z^6}{(2x+3y)^6\cos^4\frac{2\pi}{9}} > \frac{4}{3^3\cdot 5^6}.$$

Daniel Sitaru

407. Let a, b, c be the sides lengths of an acute triangle ABC. Prove that:

$$a \cdot \cos B \cdot \cos C + b \cdot \cos C \cdot \cos A + c \cdot \cos A \cdot \cos B < \sqrt{\frac{2}{3}(a^2 + b^2 + c^2)}.$$

George Apostolopoulos

408. Prove that if $P \in Int\ (\Delta ABC); PA = x; PB = y; PC = z$ then:

$$(ax + by + cz)^3(ax + by - cz)(ax - by + cz)(by + cz - ax) \leq$$
$$\leq 27(abcxyz)^2$$

Daniel Sitaru

409. Let ABC be a triangle such that $\cos(2A) + \cos(2B) + \cos(2C) + 1 = 0$, and let a, b, c and R be the edge-lenghts and the circumradius of the triangle ABC. Prove that:

$$R \geq \frac{\sqrt{2(a^2 + b^2 + c^2)}}{4}.$$

George Apostolopoulos

410. Prove that if $P \in Int(\Delta ABC); PA = x; PB = y; PC = z$ then:

$$27(by + cz - ax)(ax + cz - by)(ax + by - cz) \leq (ax + by + cz)^3.$$

Daniel Sitaru

411. Let ABC denote a triangle, I its incenter, s its semiperimeter and R_a, R_b and R_c the circumradii of triangles IBC, ICA, and IAB, respectively. Prove that:

a. $\frac{a}{R_a} + \frac{b}{R_b} + \frac{c}{R_c} \leq 3\sqrt{3}$;

b. $R_a + R_b + R_c \geq \frac{2s\sqrt{3}}{3}$.

George Apostolopoulos

412. Prove that if $P \in Int\ (\Delta ABC); PA = x; PB = y; PC = z$ then:

$$(a^2x^2 + b^2y^2 + c^2z^2)^2 \geq$$
$$\geq 3(ax + by + cz)(ax + by - cz)(ax - by + cz)(by + cz - ax).$$

Daniel Sitaru

413. Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC. Prove that

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{R}{2r^2}$$

where r and R are inradius and circumradius of ABC, respectively.

George Apostolopoulos

414. Prove that if $P \in Int(\Delta ABC); PA = x; PB = y; PC = z$ then:

$$\frac{a^2x^2 + b^2y^2 + c^2z^2}{a^2b^2c^2x^2y^2z^2} \leq$$
$$\leq \frac{9}{(ax + by + cz)(ax + by - cz)(ax - by + cz)(by + cz - ax)}.$$

Daniel Sitaru

415. Let a, b, c be the lengths of the sides of a triangle, and let r be the inradius of the triangle. Prove that:

$$3\sqrt[3]{a^2b^2c^2}+\sqrt{3(a^4+b^4+c^4)}\geq 72r^2.$$

George Apostolopoulos

416. Prove that if $P\in Int(\Delta ABC); PA=x; PB=y; PC=z$ then:

$$(ax+by+cz)^3(ax+by-cz)^3(ax-by+cz)^3(by+cz-ax)^3\leq$$
$$\leq 27(abcxyz).$$

Daniel Sitaru

417. Let n be a positive integer. Prove that in any acute triangle ABC

$$\frac{\cos^n\left(\frac{A}{2}\right)}{\sin^2\left(\frac{A}{2}\right)+\cos^n\left(\frac{A}{2}\right)}+\frac{\cos^n\left(\frac{B}{2}\right)}{\sin^2\left(\frac{B}{2}\right)+\cos^n\left(\frac{B}{2}\right)}+\frac{\cos^n\left(\frac{C}{2}\right)}{\sin^2\left(\frac{C}{2}\right)+\cos^n\left(\frac{C}{2}\right)}\leq\frac{3s^{\frac{n}{3}}}{r^{\frac{n}{3}}+s^{\frac{n}{3}}}$$

where s and r denote the semiperimeter and the inradius of ΔABC, respectively.

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418. Prove that in any ABC triangle the following relationship holds:

$$9\sum\frac{a^5}{(b+c-a)^2}\geq 8s(s_ar_a+s_br_b+s_cr_c)$$

where s_a, s_b, s_c are simedians lengths in ΔABC.

Daniel Sitaru

419. Let ABC be a triangle with $a=BC, b=CA, c=AB$ and R be its circumradius. Prove that:

$$\frac{b+c-a}{\cos\frac{A}{2}}+\frac{c+a-b}{\cos\frac{B}{2}}+\frac{a+b-c}{\cos\frac{C}{2}}\leq 6R.$$

George Apostolopoulos

420. Prove that in any acute-angled ABC triangle the following relationship holds:

$$(\tan A\tan B-\cot A\cot B)\cdot(\tan B\tan C-\cot B\cot C)\cdot$$
$$\cdot(\tan C\tan A-\cot C\cot A)\geq\frac{512}{27}.$$

Daniel Sitaru

421. Let ABC be a triangle with $a=BC, b=AC, c=AB$, and S, G be the symmedian point and the centroid point respectively. Let R, r be the circumradius and inradius respectively of ΔABC. Prove that:

$$a(b+c)\left(\frac{AS}{AG}\right)^2+b(c+a)\left(\frac{BS}{BG}\right)^2+c(a+b)\left(\frac{CS}{CG}\right)^2\geq 18\frac{(2r)^6}{R^4}.$$

George Apostolopoulos

422. Prove that if $a, b, c > 0$ then:

$$\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}} \leq \sqrt{7\left(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\right)}.$$

Daniel Sitaru

423. Prove that in any triangle ABC

a. $\frac{a^2}{w_b w_c} + \frac{b^2}{w_c w_a} + \frac{c^2}{w_a w_b} \geq 4$;

b. $\left(\frac{a}{w_b w_c}\right)^2 + \left(\frac{b}{w_c w_a}\right)^2 + \left(\frac{c}{w_a w_b}\right)^2 \geq \left(\frac{4}{3R}\right)^2$,

where R is the circumradius of ΔABC and w_a is the length of the internal bisector of angle A opposite side a.

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424. Prove that if $x, y, z > 0$ then:

$$\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}} \leq \sqrt{6\left(\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x}\right)}.$$

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425. Prove that in any triangle ABC

$$\sum \frac{\sin^n \frac{A}{2}}{\sin^n \frac{A}{2} + \cos^n \frac{A}{2}} \leq \frac{3 \cdot 2^{-\frac{n}{3}} R^{\frac{n}{3}}}{r^{\frac{n}{3}} + s^{\frac{n}{3}}},$$

for each n positive integer, where s, r, and R are the semiperimeter, the inradius, and the circumradius respectively of the triangle ABC. The sum is over all cyclic permutation of (A, B, C).

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426. Prove that in any ABC triangle the following relationship holds:

$$3\sqrt{\frac{m_a}{m_b} + \frac{2m_b}{m_c} + \frac{6m_c}{m_a}} \geq \sqrt{\frac{h_a}{m_b}} + 2\sqrt{\frac{h_b}{m_c}} + 6\sqrt{\frac{h_c}{m_a}}.$$

Daniel Sitaru

427. Let ABC be an acute triangle with sides a, b, c with $a < b < c$ opposite the angles A, B, C (measured in radians) respectively. Prove that:

$$2\left(a^2 \cdot \frac{a-b}{A-B} + b^2 \cdot \frac{b-c}{B-C} + c^2 \cdot \frac{c-a}{C-A}\right)^2 < (a^2 + b^2 + c^2)^3.$$

George Apostolopoulos

428. Prove that in any ABC triangle the following relationship is true:

$$2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right) \leq \frac{3}{\sqrt[3]{(b+c-a)(c+a-b)(a+b-c)}}.$$

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429. In all acute triangles ABC holds:

$$\frac{\sin^3 A + \sin^3 B + \sin^3 C}{\cos A \cdot \cos B \cdot \cos C} \geq 9\sqrt{3}.$$

George Apostolopoulos

430. Prove that if $a \neq b$ in ABC triangle then the following relationship is true:

$$\frac{\left(2b + 2c - 3\sqrt[3]{abc}\right)\left(1 + \left(\sqrt{a} - \sqrt{b}\right)^2\right)}{\left(\sqrt{a} - \sqrt{b}\right)^2\left(1 + a + b + c - 3\sqrt[3]{abc}\right)} > 1.$$

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431. a. Let ABC be a triangle. Prove that:

$$\sin A + \sin B + \sin C - \sin A \cdot \sin B \cdot \sin C < 2.$$

b. Let $ABCD$ be a convex quadrilateral. Prove that:

$$\sin A + \sin B + \sin C + \sin D - \sin A \cdot \sin B \cdot \sin C \cdot \sin D \leq 3.$$

George Apostolopoulos

432. Prove that in any ABC acute-angled triangle the following relationship is true:

$$A^2 + B^2 + C^2 + \cos A + \cos B + \cos C + \ln|\cos A \cos B \cos C| < 3.$$

Daniel Sitaru

433. Let a, b and c be the lengths of the sides of an acute triangle ABC with circumradius R. Prove that:

$$8R^6 - (ab + bc + ca)R^4 + \left(\frac{abc}{4}\right)^2 > 0.$$

George Apostolopoulos

434. Prove that if a, b, c are the length sides in ΔABC then:

$$\sqrt{a + b} + \sqrt{a + 2b + 2\sqrt{b(a + b - c)}} > \sqrt{2a + b + 2\sqrt{ac}}.$$

Daniel Sitaru

435. Prove that in any triangle ABC:

$$\sqrt{(\sin A + \sin B + \sin C)(\sin^3 A + \sin^3 B + \sin^3 C)} < \frac{9}{2}.$$

George Apostolopoulos

436. Let N be the Nagel point in ΔABC. Prove that:

$$\left(3\prod AN - \sum AN^3\right)^2 \leq \left(\sum a^2 + 12r^2 - 4\sqrt{3}F\right)^3.$$

Daniel Sitaru

437. Let a, b and c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that:

$$a\sqrt{\frac{b}{c}} + b\sqrt{\frac{c}{a}} + c\sqrt{\frac{a}{b}} \leq \frac{3\sqrt{6}}{2}R\sqrt{\frac{R}{r}}.$$

George Apostolopoulos

438. Prove that if in ABC triangle $a \neq b \neq c \neq a$ then:

$$\frac{2s+a}{(b-a)(c-a)\sin A}+\frac{2s+b}{(c-b)(a-b)\sin B}+\frac{2s+c}{(a-c)(b-c)\sin C}>\frac{2}{R}.$$

Daniel Sitaru

439. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r. Prove that:

$$\frac{\tan\frac{A}{2}}{a^3}+\frac{\tan\frac{B}{2}}{b^3}+\frac{\tan\frac{C}{2}}{c^3}\leq\frac{1}{24r^3}.$$

George Apostolopoulos

440. Prove that in any ABC acute-angled triangle the following relationship holds:

$$2F^2\sum(\sin A+\cos A+\tan A+\cot A)>81\pi R^4\prod\cos A.$$

Daniel Sitaru

441. Let ABC be a triangle and let x, y, w be positive real numbers such that $x^4+y^4+w^4=6xyw$. Prove that:

$$x\cdot\cos A+y\cdot\cos B+w\cdot\cos C\leq 3.$$

George Apostolopoulos

442. Prove that if $x, y, z > 0$ then:

$$\sqrt{x^2-xy\sqrt{3}+y^2}+\sqrt{y^2-yz\sqrt{3}+z^2}+\sqrt{z^2-zx\sqrt{3}+x^2}\geq\frac{\sqrt{2}}{3}(x+y+z).$$

Daniel Sitaru

443. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\cot A+\cot B+\cot C\leq\sqrt{3}\cdot\left(\frac{R}{2r}\right)^2.$$

George Apostolopoulos

444. Prove that in any ABC triangle the following relationship holds:

$$R(\sin A\sin 5A+\sin B\sin 5B+\sin C\sin 5C)<10(2R-r).$$

Daniel Sitaru

445. Let ABC be a triangle. Prove that:

$$\frac{(\sin A+\sin B)^2}{\sin C}+\frac{(\sin B+\sin C)^2}{\sin A}+\frac{(\sin C+\sin A)^2}{\sin B}\leq\frac{3\sqrt{3}}{4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}.$$

George Apostolopoulos

446. Prove that in an ABC acute-angled triangle the following relationship holds:

$$\sum\frac{1}{\sin 2B+\sin 2C-\sin 2A}\geq\frac{3}{\sqrt[3]{\sin 2A\sin 2B\sin 2C}}\geq 2\sqrt{3}.$$

Daniel Sitaru

447. For a triangle ABC, prove that:

$$9\sqrt{3}\prod \cos A \leq \sum \sin^3 A \leq \frac{3\sqrt{3}}{2}\sum \sin^2 \frac{A}{2}$$

where the sums and product are taken all the angles of the triangle.

George Apostolopoulos

448. Prove that in an ABC acute-angled triangle the following relationship holds:

$$\frac{a^3\cos^2 A}{\arctan\frac{1}{2}}+\frac{b^3\cos^3 B}{\arctan\frac{1}{5}}+\frac{c^3\cos^3 C}{\arctan\frac{1}{8}}\geq\frac{32r^3s^3}{3\pi R^3}.$$

Daniel Sitaru

449. Let r, r_a, r_b, r_c be, respectively, the inradius and the exradii of a triangle ABC with side lengths a, b, c. Prove that:

$$\frac{(r\cdot r_a)^2}{a^4}+\frac{(r\cdot r_b)^2}{b^4}+\frac{(r\cdot r_c)^2}{c^4}\leq\frac{3}{16}.$$

George Apostolopoulos

450. Prove that in an ABC acute-angled triangle the following relationship holds:

$$a^4\cos^3 A\sin B\sin C+b^4\cos^3 B\sin C\sin A+c^4\cos^3 C\sin A\sin B\geq\frac{8r^4s^4}{9R^4}.$$

Daniel Sitaru

451. Let a, b, c be the side lengths of a triangle ABC with inradius r, and circumradius R. Prove that:

$$\frac{1}{a^3}\tan\frac{A}{2}+\frac{1}{b^3}+\tan\frac{B}{2}+\frac{1}{c^3}\tan\frac{C}{2}\leq\frac{R}{48r^4}.$$

George Apostolopoulos

452. Prove that in any ABC triangle the following relationship holds:

$$\frac{ab}{c(a+b)^2}+\frac{bc}{a(b+c)^2}+\frac{ca}{b(c+a)^2}\geq\frac{\sqrt{3}}{4R}.$$

Daniel Sitaru

453. Let r, r_a, r_b, r_c be, respectively, the inradius, and the exradii of a triangle ABC with side lengths a, b, c. Prove that:

a. $\dfrac{b+c}{\sqrt{r\cdot r_a}}+\dfrac{c+a}{\sqrt{r\cdot r_b}}+\dfrac{a+b}{\sqrt{r\cdot r_c}}\geq 12;$

b. $\dfrac{(r\cdot r_a)^n}{a^{2n}}+\dfrac{(r\cdot r_b)^n}{b^{2n}}+\dfrac{(r\cdot r_c)^n}{c^{2n}}\leq\dfrac{3}{4^n}$, for each integer $n\geq 0$.

George Apostolopoulos

454. Prove that in any ABC acute-angled triangle the following relationship holds:

$$2\left(\sum \cot A\cot B\right)\left(\sum\frac{\cot A\tan B\tan C}{\tan B+\tan C}\right)\geq\left(\sum\cot A\right)^2.$$

Daniel Sitaru

455. Let a, b and c be the side lengths of a triangle ABC, with circumradius R and inradius r. Prove that:

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{a+b}+\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{b+c}+\frac{\tan\frac{C}{2}+\tan\frac{A}{2}}{c+a}\leq\frac{1}{r}-\frac{1}{R}.$$

George Apostolopoulos

456. Prove that in any ABC triangle the following relationship holds:

$$\frac{1}{R}\sum\frac{m_c}{a+b}<\sum\frac{m_c}{aR+br}<\frac{1}{2R}\left(3+\sum\frac{b}{a}\right).$$

Daniel Sitaru

457. Let a, b, c be positive real numbers such that $abc=1$, and let x, y, z be real numbers such that $xy+yz+zx\geq 3$. Prove that:

$$(3x^2+10)\cdot\frac{a^3+b^3}{a^2+ab+b^2}+(3y^2+10)\cdot\frac{b^3+c^3}{b^2+bc+c^2}+$$
$$+(3z^2+10)\cdot\frac{c^3+a^3}{c^2+ca+a^2}\geq 26.$$

George Apostolopoulos

458. Prove that in any ABC triangle the following relationship holds:

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\leq 3R\sqrt{2s}.$$

Daniel Sitaru

459. In a triangle ABC, points D and E respectively divide the sides AB and AC in the ratios $\frac{AD}{BD}=m$, and $\frac{AE}{CE}=n$. The segments BE and CD intersect in a point O. Prove that $[DOE]\leq\frac{\sqrt[4]{m\cdot n}}{4}([BOD]+[COE])$, where $[XYZ]$ represents the area of triangle XYZ.

George Apostolopoulos

460. Prove that in any ABC triangle the following relationship holds:

$$\sum\frac{\sin^8 A+\sin^4 A+\cos^8 A+\cos^4 A}{\sin^6 A+\sin^4 A+\cos^4 A+\cos^6 A}\geq\frac{12r}{R}.$$

Daniel Sitaru

461. In a triangle ABC, let l_a, l_b, l_c be the lengths of the internal angle bisectors, let m_a, m_b, m_c be the lengths of the medians, and let R be the circumradius. Prove that:

$$\frac{l_a^2}{m_a}+\frac{l_b^2}{m_b}+\frac{l_c^2}{m_c}\leq\frac{9}{2}R.$$

George Apostolopoulos

462. Prove that in any ABC triangle the following relationship holds:

$$\sum\left(\frac{am_a}{m_b}+b\right)^2+\sum\left(\frac{am_b}{m_a}+b\right)^2\geq 4(3s^2-r^2-4Rr).$$

Daniel Sitaru

463. Let ABC be a triangle and (I, r) its incircle. The circle (I_a, r_a) is externally tangent to the circle (I, r) and internally tangent to the sides AB and AC of the triangle. The circles (I_B, r_b) and (I_C, r_c) are defined similarly. Prove that:

$$\left(\frac{r+r_a}{r-r_A}\right)^n+\left(\frac{r+r_b}{r-r_B}\right)^n+\left(\frac{r+r_c}{r-r_C}\right)^n \geq 3(n+1)$$

for each natural number n.

George Apostolopoulos

464. Prove that in ABC triangle the following relationship holds:

$$\sum \sqrt{a^2+(2s-a)^2+2a(2s-a)\cos A} < 6s.$$

Daniel Sitaru

465. Let ABC be a triangle and (I, r) its incircle. The circle (I_A, r_A) is externally to the circle (I, r) and internally tangent to the sides AB and AC of the triangle. The circles (I_B, r_B) and (I_c, r_c) are defined similarly. Prove that:

a. $r_A+r_B+r_C \geq r$;

b. $\dfrac{(r-r_A)^2}{rr_A}+\dfrac{(r-r_B)^2}{rr_B}+\dfrac{(r-r_C)^2}{rr_C} \geq 4.$

George Apostolopoulos

466. Prove that in any ABC triangle the following relationship holds:

$$2\sum a^4 < 2\sum b^2c^2\cos 2A+\sum a^2\,(b+c)^2.$$

Daniel Sitaru

467. The incircle of the right triangle ABC ($A=90°$) touches the sides BC, CA, AB at the points L, M, N, respectively. Prove that:

$$\left(\frac{LM}{AB}\right)^4+\left(\frac{LN}{AC}\right)^4 \geq 2\left(\frac{MN}{BC}\right)^2.$$

George Apostolopoulos

468. Prove that if $x, y, z \in [0, \infty)$ then

$$\sqrt{x^2-xy\sqrt{3}+y^2}+\sqrt{y^2-yz\sqrt{2}+z^2} \geq \sqrt{x^2-xz+z^2}.$$

Daniel Sitaru

469. Let $ABCD$ be a convex quadrilateral and O be the intersection point of the diagonals, and let K and L be points on side CD such that $OK \parallel AD$ and $OL \parallel BC$. If $DK^2+LC^2=KL^2$, then determine the ratio of the areas of $ABCD$ and OCD.

George Apostolopoulos

470. Prove that in any ABC triangle the following relationship holds:

$$3\sum \frac{1}{a^2+ab+b^2} \leq \frac{1}{6F}\sum \sin A.$$

Daniel Sitaru

471. Let ABC be a triangle such that $\sphericalangle B = 2\sphericalangle C$. We extend the side BC by a segment CD equal to $\frac{1}{3}BC$. Prove that:

$$Area(ABC) = \frac{1}{4}|BC|^2 \cdot \cot\frac{\theta}{2}, \text{ where } \theta = \sphericalangle BAD.$$

George Apostolopoulos

472. Prove that in any ABC acute-angled triangle the following relationship holds:

$$\prod (a^{\cos A}) \cdot (\cos A)^a \le \frac{(\sum \cos A)^{2s} \cdot (2p)^{\sum \cos A}}{3^{2s+\sum \cos A}}.$$

Daniel Sitaru

473. Triangle ABC is isosceles with $AB = AC$ and $\sphericalangle A = 100°$. Let D be the point on AB such that $\sphericalangle BCD = 10°$ and let E be the point on BC such that $EC = AC$. Determine the point K on CD such that triangles KAD and KCE have equal areas.

George Apostolopoulos

474. Prove that in ABC triangle $a \ne b \ne c \ne a$ then:

$$\sum \frac{a+b}{(c-a)(c-b)\sin C} \ge \frac{2}{R}.$$

Daniel Sitaru

475. Given a square $ABCD$ with side length a. Points K and L are on BC and CD, respectively, such that the perimeter of ΔKCL is a $2a$. Determine the measures of the angles of ΔAKL which minimize its area.

George Apostolopoulos

476. In ABC triangle let be ω – Brocard angle and Ω – first Brocard point of ΔABC. Prove that:

$$a^2 + b^2 + c^2 \le 4F \cot\omega \left(\frac{\Omega A \cdot \Omega B}{ab} + \frac{\Omega B \cdot \Omega C}{bc} + \frac{\Omega C \cdot \Omega A}{ca}\right).$$

Daniel Sitaru

477. Let ABC be a triangle with incentre I through which an arbitrary line passes meeting sides AB and AC at the points D and E respectively. Show that:

$$\frac{1}{r} \ge \frac{1}{AD} + \frac{1}{AE}.$$

George Apostolopoulos

478. Prove that in any ABC triangle the following relationship holds:

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \ge \sqrt{4 + \frac{2\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{F}}.$$

Daniel Sitaru

479. Let ABC be a triangle with incentre I through which an arbitrary line passes meeting sides AB and AC at the points D and E respectively. Show that:

$$\frac{[ABC]^2}{[ADE]} \leq \frac{27}{4} \cdot \frac{R^3}{BC},$$

where R and $[XYZ]$ are the circumradius of ΔABC and the area of ΔXYZ respectively.

George Apostolopoulos

480. Prove that in any ABC triangle the following relationship holds:

$$a^6 + b^6 + c^6 \geq 8r^2 s \sum \frac{a^5}{b^2 - bc + c^2}.$$

Daniel Sitaru

481. Let ABC be an isosceles triangle with $AB = AC$ and $\sphericalangle A = 100°$. On side AB we get the inner point D so that $\sphericalangle BCD = 10°$ and on the side BC we get inner point E so that $EC = AC$. Find an inner point K of the segment CD so that KAD and KCE triangles have equal areas.

George Apostolopoulos

482. If $a, b, c > 0$ then the following relationship holds:

$$\int_a^{2a} \left(\int_b^{2b} \left(\int_c^{2c} \left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right) dx \right) dy \right) dz \leq \ln \sqrt{2^{ab+bc+ca}}.$$

Daniel Sitaru

483. Let ABC be a triangle. Prove that:

$$\sin\frac{A}{2} \cdot \sin B \cdot \sin C + \sin A \cdot \sin\frac{B}{2} \cdot \sin C + \sin A \cdot \sin B \cdot \sin\frac{C}{2} \leq \frac{9}{8}.$$

George Apostolopoulos

484. If $a, b, c > 0$ then the following relationship holds:

$$\frac{a}{2017a + b + c} + \frac{b}{a + 2017b + c} + \frac{c}{a + b + 2017c} \leq \frac{3}{2019}.$$

Daniel Sitaru

485. For an acute triangle ABC and a positive integer n, prove that:

$$\left(\sum (\sin A \cdot \sin B \cdot \sin C)^{\frac{1}{n}} \right)^n \leq \frac{3^{n+1}}{8},$$

where the sum is over all cyclic permutations of (A, B, C).

George Apostolopoulos

486. In ΔABC the following relationship holds:

$$\sum c^2 (1 + a + b + ab) \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \right) \geq 64\sqrt{3}F.$$

Daniel Sitaru

487. Let a, b and c be the lengths of the sides of a triangle ABC with circumradius R and inradius r. Prove that:

$$\frac{R}{r} \geq \frac{2}{3}(\cos A + \cos B + \cos C) + \frac{a^3 + b^3 + c^3}{3abc}.$$

George Apostolopoulos

488. If $1 < x < y$ then the following relationship holds:

$$x^{\frac{1}{x-1}} > y^{\frac{1}{y-1}}.$$

Daniel Sitaru

489. For an acute triangle ABC prove that:

$$\sum \frac{\sec A}{\sqrt{\cos A + \cos B}} \geq 6,$$

where the sum Σ is over all cyclic permutations of (A, B, C).

George Apostolopoulos

490. In ΔABC the following relationship holds:

$$\prod \left(\frac{b^2}{\sin^2 \frac{B}{2}} + \frac{c^2}{\sin^2 \frac{C}{2}} - \frac{a^2}{\sin^2 \frac{A}{2}} \right) \leq \prod \left(\frac{a}{\sin \frac{A}{2}} \right)^2.$$

Daniel Sitaru

491. In all triangle ABC holds:

$$160 \cdot \sum_{cyc} \left(\frac{A}{\pi} \right)^2 - 27 \sum_{cyc} \left(1 - 2\frac{A}{\pi} \right)^5 \geq 53.$$

George Apostolopoulos

492. In ΔABC the following relationship holds:

$$4F\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{2m_c}\right) < \sqrt{3}(a + b + 2m_c).$$

Daniel Sitaru

493. In all triangle ABC holds:

a. $\dfrac{4s^2 r}{R} \leq \dfrac{h_a^3 + h_b^3}{r_c} + \dfrac{h_b^3 + h_c^3}{r_a} + \dfrac{h_c^3 + h_a^3}{r_b} \leq 2s^2$, where $s = \dfrac{a + b + c}{2}$;

b. $\dfrac{3}{2R} \leq \dfrac{r_a}{a^2} + \dfrac{r_b}{b^2} + \dfrac{r_c}{c^2} \leq \dfrac{3}{4r}$.

George Apostolopoulos

494. If $f: [a, b] \to \mathbb{R}, ab \geq 0,\ f$ – continuous, f – increasing, then:

$$\left(\sqrt{a} + \sqrt{b}\right) \int_a^{\sqrt{ab}} f(x)dx \leq \sqrt{a} \int_a^b f(x)dx.$$

Daniel Sitaru

495. Let a, b and c be the lengths of the sides of a triangle, let R and s be the circumradius and the semiperimeter, respectively. Prove that:

$$\left(\frac{a}{b}\right)^n + \left(\frac{b}{c}\right)^n + \left(\frac{c}{a}\right)^n \geq 3^{1-n} \cdot \left(\frac{2s}{R}\right)^{\frac{2n}{3}},$$

for each natural number n.

George Apostolopoulos

496. If $a, b, c, d > 0$ then the following relationship holds:

$$\sum a \arctan a \geq \sum \sqrt[3]{bcd} \arctan a \geq 4\sqrt[4]{abcd \prod \arctan a}.$$

Daniel Sitaru

497. Prove that in all non-obtuse angled triangle ABC holds:

$$\sum_{cyc} (\sin A \cdot \sin B \cdot \cos C)^{\frac{1}{2}} \leq \frac{3\sqrt{6}}{4}.$$

George Apostolopoulos

498. If $a, b, c > 0, a^2 + b^2 + c^2 = 3$ then the following relationship holds:

$$\sum (a+b)c^2 + \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 9.$$

Daniel Sitaru

499. Prove that in any triangle ABC:

$$27(\cos A + \cos B + \cos C) - 4(\cos^3 A + \cos^3 B + \cos^3 C) \leq 39.$$

George Apostolopoulos

500. If $a, b, c > 0$ then:

$$4ab \cdot \arctan\left(\frac{c}{b}\right) + 4bc \cdot \arctan\left(\frac{a}{c}\right) + 4ca \cdot \arctan\left(\frac{b}{a}\right) \leq \pi(ab + bc + ca).$$

Daniel Sitaru

501. Let ABC be an acute triangle with circumradius R, inradius r, and semiperimeter s. Prove that: $s^2 > r(8R + r)$.

George Apostolopoulos

502. If $a, b, c > 0, a^2 + b^2 + c^2 = 3$ then the following relationship holds:

$$\sum (a+b)\sqrt{a(a+b)} + \sum b\sqrt{a^2 + b^2} \leq \sqrt{2}(6 + ab + bc + ca).$$

Daniel Sitaru

503. Let ABC be a triangle where R and r are the circumradius and the inradius respectively. Prove that:

$$(\cos A - \cos B - \cos C)^3 - (\cos A - \cos B + \cos C)^3 -$$
$$-(\cos A + \cos B - \cos C)^3 + 21\left(\frac{r}{R}\right)^3 \leq \frac{9}{4}.$$

George Apostolopoulos

504. If $x, y, z \in \mathbb{R}, a > 0, |x| \leq a, |y| \leq a, |z| \leq a$ then:

$$\sqrt{7(a^2 - x^2)} + \sqrt{7(a^2 - y^2)} + \sqrt{7(a^2 - z^2)} + 9\sqrt[3]{xyz} \leq 12a.$$

Daniel Sitaru

505. Let ABC be an acute triangle. Show that:

$$\sum \frac{\cos A \cdot \cos B}{\cos C} \geq \frac{3}{2},$$

where the sum Σ is over all cyclic permutations of (A, B, C).

George Apostolopoulos

506. If $a, b, c, x, y, z > 0, a + b + c \geq 3$ then the following relationship holds:

$$(ax + by + cz)(ay + bz + cx)(az + bx + cy) \geq \frac{729}{\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^3}.$$

Daniel Sitaru

507. Given an isosceles triangle ABC with $\measuredangle BAC = 100°$. On the side AB and towards the direction of the point B we select a point D such that $AD = BC$. Prove that:

a. $\frac{CD}{BC} = 2\sin 40°$; b. $BC^3 < AC \cdot CD^2$.

George Apostolopoulos

508. Find:

$$\Omega = \int \frac{\cos 2x \cot x \, dx}{(\cot^2 x - \tan^2 x)\sin^3 2x}, x \in \left(0, \frac{\pi}{4}\right).$$

Daniel Sitaru

509. A square $ABCD$ with side length a is given. On the sides BC, CD we consider interior points K, L respectively so that the perimeter of triangle KCL has to be equal to $2a$. If the diagonal BD intersects the segments AK, AL at the points N, M respectively, prove that $Area\ \Delta AMN = Area\ KLMN$.

George Apostolopoulos

510. If $0 < a < b; 0 < c < d; f, g$ integrable functions $f, g: [a, b] \to [c, d]$ then:

$$cd\left(\int_a^b \frac{f(x)}{g(x)}dx + \int_a^b \frac{g(x)}{f(x)}dx\right) < (c^2 + d^2)(b - a).$$

Daniel Sitaru

511. On sides BC, CA and AB of a triangle ABC points A_1, B_1 and C_1 respectively, are taken such that:

$$\frac{A_1B}{A_1C} = \frac{B_1C}{B_1A} = \frac{C_1A}{C_1B} = l.$$

Prove that:

$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{CA}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \left(\frac{3l}{l^2 + 1}\right)^2 \cdot \left(\frac{2r}{R}\right)^4,$$

where R, r are the circumradius and inradius of the triangle ABC respectively.

George Apostolopoulos

512. If $x, y, z \geq 1$ then:

$$\frac{1}{1+\sqrt{(x-1)(y-1)}}+\frac{1}{1+\sqrt{(y-1)(z-1)}}+\frac{1}{1+\sqrt{(z-1)(x-1)}} \geq \frac{3}{\sqrt[3]{xyz}}.$$

Daniel Sitaru

513. Let a, b and c be the lengths of the sides of a triangle ABC with $\measuredangle C = 2\measuredangle B$. On the side BC consider a point D such that $\measuredangle BAD = \frac{1}{4}\measuredangle C$. Prove that:

a. $\frac{[ADC]}{[ABD]} = \frac{ab-bc+ca}{bc}$, where $[XYZ]$ represents the area of $\Delta\ XYZ$;

b. $AD = \frac{c}{b+c}\sqrt{2b^2+bc}$.

George Apostolopoulos

514. If $a, b, c, d > 0, a+b+c+d = abcd$ then the following relationship holds:

$$(a^3+b^3+c^3+d^3)a^3b^3c^3d^3 \geq 4096.$$

Daniel Sitaru

515. Let ABC be a triangle with circumradius R and inradius r, and let w_a, w_b, w_c be the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c, respectively. Prove that:

$$\left(\frac{w_a}{a}\right)^2 \cdot \tan\frac{A}{2} + \left(\frac{w_b}{b}\right)^2 \cdot \tan\frac{B}{2} + \left(\frac{w_c}{c}\right)^2 \cdot \tan\frac{C}{2} \leq \frac{3\sqrt{3}}{8} \cdot \frac{R}{r}.$$

George Apostolopoulos

516. If $0 \leq x, y, z < 1$ then the following relationship holds:

$$\sqrt[3]{\frac{(1+x^3)(1+y^6)(1+z^9)}{(1-x^3)(1-y^6)(1-z^9)}} \geq \frac{1+xy^2z^3}{1-xy^2z^3}.$$

Daniel Sitaru

517. Let a, b and c denote, as usual, the lengths of the side BC, CA and AB, respectively, in ΔABC. Let R be the circumradius, r the inradius of ΔABC, and r_a, r_b and r_c the exradii to A, B and C, respectively. Prove that:

a. $\frac{r_a}{a^3}+\frac{r_b}{b^3}+\frac{r_c}{c^3} \leq \frac{\sqrt{3}}{8r^2}$;

b. $\frac{3R}{2r} \geq \sqrt{\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6}$.

George Apostolopoulos

518. If $a, b, c, x, y, z \geq 0, a+b+c = 1$ then the following relationship holds:

$$\frac{1}{1+x^ay^bz^c}+\frac{1}{1+x^by^cz^a}+\frac{1}{1+x^cy^az^b} \geq \frac{9}{3+x+y+z}.$$

Daniel Sitaru

519. Let a, b, c be the lengths of the sides of a triangle ABC with circumradius R and inradius r. Prove that:

$$\sqrt[3]{\frac{a}{b+c-a}}+\sqrt[3]{\frac{b}{c+a-b}}+\sqrt[3]{\frac{c}{a+b-c}}\le\frac{3R}{2r}.$$

George Apostolopoulos

520. If $x, y, z > 0$ then:

$$\frac{5x+3y+z}{5z+3y+x}+\frac{5y+3z+x}{5x+3z+y}+\frac{5z+3x+y}{5y+3x+z}\ge 3.$$

Daniel Sitaru

521. Let a, b, c be the side lengths of a triangle ABC, and let r_a, r_b, r_c be the exradii of triangle ABC with inradius r and circumradius R. Prove that:

a. $\frac{3}{2R}\le\frac{r_a}{a^2}+\frac{r_b}{b^2}+\frac{r_c}{c^2}\le\frac{3}{4r}$;

b. $\frac{1}{2R^3}\le\frac{r_a}{a^4}+\frac{r_b}{b^4}+\frac{r_c}{c^4}\le\frac{1}{16r^3}$.

George Apostolopoulos

522. If $a, b, c \in \mathbb{R}, a+b+c=0$ then the following relationship holds:

$$5\left(\sum a^3\right)\left(\sum a^7\right)\ge 39abc\sqrt[3]{a^2b^2c^2}\left(\sum a^5\right).$$

Daniel Sitaru

523. Let a, b, c be the lengths of the sides of a triangle ABC with circumcenter O, circumradius R and inradius r. Let x, y, z be the distances from O to the sides $\overline{BC}, \overline{AC}, \overline{AB}$ respectively. Prove that:

$$\frac{z}{a}+\frac{x}{b}+\frac{y}{c}\le\frac{\sqrt{3}}{8}\left(\frac{R}{r}\right)^2.$$

George Apostolopoulos

524. If $x > 0$ then the following relationship holds:

$$\left(1+\frac{x}{5}\right)^{20}>(1+x)\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{3}\right)^3\left(1+\frac{x}{4}\right)^4.$$

Daniel Sitaru

525. Let $ABCD$ be a cyclic quadrilateral with the lengths of the sides a, b, c, d. Prove that:

$$\frac{a^5+b^5+c^5}{a^3+b^3+c^3}+\frac{b^5+c^5+d^5}{b^3+c^3+d^3}+\frac{c^5+d^5+a^5}{c^3+d^3+a^3}+\frac{d^5+a^5+b^5}{d^3+a^3+b^3}\ge 4[ABCD],$$

where $[ABCD]$ represents the area of $ABCD$.

George Apostolopoulos

526. If $x, y, z \in \mathbb{R}, x^4+y^4+z^4=3$ then the following relationship holds:

$$\sum(2x+y)^4-7\sum(x+y)^4+\sum(x+2y)^4\le 150.$$

Daniel Sitaru

527. Let ABC be a triangle with circumradius R and inradius r, and let w_a, w_b, w_c be the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c, respectively. Prove that:

$$\frac{3}{8}\cdot\frac{a^2+b^2+c^2}{w_a^2+w_b^2+w_c^2}\geq\frac{r}{R}.$$

George Apostolopoulos

528. In ΔABC:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}>\sum\frac{\sin\frac{A}{3}}{\sin\left(C+\frac{A}{3}\right)}.$$

Daniel Sitaru

529. Let ABC be a triangle such that:

$$\left(\frac{1}{\sin B}+\frac{1}{\sin C}\right)(-\sin A+\sin B+\sin C)=2.$$

Prove that $\measuredangle A\leq\frac{\pi}{3}$.

George Apostolopoulos

530. If $a, b, c > 0$ then the following relationship holds:

$$\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c})}{(\sqrt[8]{a}+\sqrt[8]{b}+\sqrt[8]{c})(\sqrt[16]{a}+\sqrt[16]{b}+\sqrt[16]{c})}\geq\sqrt[16]{a^3b^3c^3}.$$

Daniel Sitaru

531. In triangle ABC, let A_1, B_1, C_1 be the points opposite A, B, C at which the angle internal bisectors of the triangle meet the opposite sides. Let R and r be the circumradius and inradius of ABC. Let a, b, c be the lengths of the sides opposite A, B, C and let a_1, b_1, c_1 be the lengths of the line segments B_1C_1, C_1A_1, A_1B_1. Prove that:

$$\left(\frac{a_1}{a}\right)^4\tan\frac{A}{2}+\left(\frac{b_1}{b}\right)^4\tan\frac{B}{2}+\left(\frac{c_1}{c}\right)^4\tan\frac{C}{2}\leq\frac{\sqrt{3}}{64}\left(\frac{R}{r}\right)^2.$$

George Apostolopoulos

532. If $x, y, z > 0$ then the following relationship holds:

$$\sum\frac{\sqrt{x}}{3\sqrt{y}+5\sqrt{z}}+\frac{\sqrt{xy}+\sqrt{yz}+\sqrt{zx}}{8(x+y+z)}\geq\frac{1}{2}.$$

Daniel Sitaru

533. Prove that in any triangle ABC the inequality:

$$\sqrt[3]{\frac{r_a}{h_a}}+\sqrt[3]{\frac{r_b}{h_b}}+\sqrt[3]{\frac{r_c}{h_c}}\leq\frac{3R}{2r}$$

holds, where r_a, r_b, r_c are the lengths of the exradii, h_a, h_b, h_c are the lengths of the altitudes, and R, r are the circumradius, inradius of ΔABC, respectively.

George Apostolopoulos

534. If $a, b, c > 0, abc = 1$ then the following relationship holds:

$$\frac{27a^9}{2+a^3} + \frac{27b^9}{2+b^3} + \frac{27c^9}{2+c^3} \geq (a^2+b^2+c^2)^3.$$

Daniel Sitaru

535. Let a, b, c be the side lengths, and let m_a, m_b, m_c be the lengths of the medians of an acute triangle ABC. Prove that:

$$\frac{m_a^2}{b^2+c^2} + \frac{m_b^2}{c^2+a^2} + \frac{m_c^2}{a^2+b^2} \geq 9 \cdot \cos A \cdot \cos B \cdot \cos C.$$

George Apostolopoulos

536. If $a, b, c > 0$ then the following relationship holds:

$$\frac{|(1-ab)(1-bc)(1-ca)|}{(1+a^2)^2(1+b^2)(1+c^2)^2} \leq \frac{1}{64abc}.$$

Daniel Sitaru

537. Let a, b, c be the lengths of the sides of a triangle ABC with circumradius R and inradius r. Prove that:

$$24\sqrt{3} \cdot r \leq \frac{(a+b)^2}{c} + \frac{(b+c)^2}{a} + \frac{(c+a)^2}{b} \leq 6\sqrt{3}\frac{R^2}{r}.$$

George Apostolopoulos

538. If $x, y, z > 0$ then the following relationship holds:

$$\prod(2x^2+xy+xz-yz) \leq \prod(x^2+xy+y^2).$$

Daniel Sitaru

539. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\frac{\cos\frac{A}{2}}{\sin^5\frac{A}{2}} + \frac{\cos\frac{B}{2}}{\sin^5\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin^5\frac{C}{2}} \geq 192\sqrt{3} \cdot \left(\frac{r}{R}\right)^2.$$

George Apostolopoulos

540. If in $\Delta ABC, a \leq b \leq c$ then the following relationship holds:

$$\frac{a^2h_a + b^2s_a + c^2m_a}{ah_a + bs_a + cm_a} \geq 2r\sqrt{3}.$$

Daniel Sitaru

541. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\frac{R}{2r} \geq \sqrt{3} \cdot \left(\frac{\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}}{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}\right)^{\frac{1}{2}}.$$

George Apostolopoulos

542. If $0 < a < b$ then the following relationship holds:

$$\int_a^b \frac{dx}{(x^3+1)^2} > \frac{5}{9(b^5-a^5)} \ln^2\left(\frac{b^3+1}{a^3+1}\right).$$

Daniel Sitaru

543. Let ABC be a triangle with circumradius R and inradius r. Prove that:

$$\frac{\cos\frac{A}{2}}{\cos\left(\frac{A}{2}-\frac{\pi}{3}\right)} + \frac{\cos\frac{B}{2}}{\cos\left(\frac{B}{2}-\frac{\pi}{3}\right)} + \frac{\cos\frac{C}{2}}{\cos\left(\frac{C}{2}-\frac{\pi}{3}\right)} \geq 12\left(\frac{r}{R}\right)^2.$$

George Apostolopoulos

544. If $x, y, z > 0$, $\frac{xy}{x+y} + \frac{yz}{y+z} + \frac{zx}{z+x} = \frac{3}{2}$ then the following relationship holds:

$$x + y + z + 3 \geq 2\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right).$$

Daniel Sitaru

545. Let ABC be a triangle. Prove that:

$$\frac{\sin\frac{A}{2}}{\cos^3\frac{C}{2}} + \frac{\sin\frac{B}{2}}{\cos^3\frac{A}{2}} + \frac{\sin\frac{C}{2}}{\cos^3\frac{B}{2}} \leq \frac{\sqrt{3}}{48 \cdot \sin^2\frac{A}{2} \cdot \sin^2\frac{B}{2} \cdot \sin^2\frac{C}{2}}.$$

George Apostolopoulos

546. Prove that:

$$\Omega = \int_0^1\int_0^1\int_0^1\int_0^1 \left(\sqrt[3]{xyz} + \sqrt[3]{yzt} + \sqrt[3]{ztx} + \sqrt[3]{txy}\right) dxdydzdt \leq 2.$$

Daniel Sitaru

547. Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC and let w_a, w_b, w_c be the lengths of the internal bisectors of the angle opposite of the sides a, b, c of the triangle ABC, respectively. Prove that:

$$\sin^2 A \cdot \frac{w_a}{m_a} + \sin^2 B \cdot \frac{w_b}{m_b} + \sin^2 C \cdot \frac{w_c}{m_c} \leq \frac{9R}{8r},$$

where R, r are the circumradius and the inradius respectively, of the triangle ABC.

George Apostolopoulos

548. If $a, b, c, d > 0$ then the following relationship holds:

$$a^a \cdot b^b \cdot c^c \cdot d^d \geq a^{\sqrt[3]{bcd}} \cdot b^{\sqrt[3]{cda}} \cdot c^{\sqrt[3]{dab}} \cdot d^{\sqrt[3]{abc}}.$$

Daniel Sitaru

549. In triangle ABC the internal bisectors of angles B and C meet the opposite sides at D and E respectively. The triangle ABC has incenter I such that $S_{AEID} = S_{BIC}$, where S denotes the area. Prove that:

a. $\sin^3 A \cdot S_{CID} = \sin^3 B \cdot S_{BIE}$; b. $\measuredangle BIC \leq \frac{2\pi}{3}$.

George Apostolopoulos

550. If in $\Delta ABC, am_a + bm_b + cm_c = 2$ then the following relationship holds:

$$s\sum am_a h_a \geq ah_a + bh_b + ch_c.$$

Daniel Sitaru

551. Prove that in all triangles ABC the inequality:

$$\frac{a^4 + b^4 + c^4}{(a+b+c)\left(\frac{a}{h_a m_a} + \frac{b}{h_b m_b} + \frac{c}{h_c m_c}\right)} \geq 36r^4,$$

where h_a, h_b, h_c and m_a, m_b, m_c are the lengths of altitudes and medians respectively, where $|AB| = c, |BC| = a, |CA| = b$ and r is the inradius of triangle ABC.

George Apostolopoulos

552. In ΔABC the following relationship holds:

$$(a^2 + b^2 + c^2)\sqrt{a^2 + b^2 + c^2} \geq 6abc\sqrt{6\cos A \cos B \cos C}.$$

Daniel Sitaru

553. Let a, b, c be the lengths of the sides of a triangle ABC with circumradius R and inradius r. Prove that:

$$12\frac{r}{R} \leq \frac{a+b}{a+b-c} + \frac{b+c}{b+c-a} + \frac{c+a}{c+a-b} \leq 3\frac{R}{r}.$$

George Apostolopoulos

554. If $a, b, c, d > 0$ then the following relationship holds:

$$\sqrt{\frac{a}{b}} + \sqrt[4]{\frac{b}{c}} + \sqrt[6]{\frac{c}{d}} + \sqrt[8]{\frac{d}{a}} > 1.$$

Daniel Sitaru

555. Let a, b, c be the lengths of the sides of a right triangle with a the length of the hypotenuse. Prove that:

$$\frac{a(a^3 + b^3 + c^3)}{b^2c^2} \geq 4 + 2\sqrt{2}.$$

George Apostolopoulos

556. In ΔABC the following relationship holds:

$$\sum \frac{a^3 + 2b^3 + 3c^3}{a + 2b + 3c} \geq 36r^2.$$

Daniel Sitaru

557. Let a, b and c be the sides of a triangle ABC with symmedian point S and w_a is the length of the internal bisector of angle A opposite side a. Prove that:

$$\left(\frac{AS}{w_a}\right)^4 + \left(\frac{BS}{w_b}\right)^4 + \left(\frac{CS}{w_c}\right)^4 \geq \frac{16}{3} \cdot \frac{a^4b^4c^4}{(a^6 + b^6 + c^6)^2}.$$

George Apostolopoulos