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THE OLYMPIC MATHEMATICAL MARATHON

grades 7-12

VOLUME 1

Cartea Românească
EDUCAȚIONAL

Dedicated to the International Mathematical Olympiad
Romania 2018

GEORGE APOSTOLOPOULOS

George Apostolopoulos was born in Messolonghi – Greece at 23 January 1949. He graduated the University of Ioannina. He is member of "The Mathematical Society of Greece" and member of the tender committee. He is also a member of "The Mathematical Society" of USA, Canada and Romania. He has published proposed mathematical problems and mathematical solutions through the scientific magazines of the "Mathematical Society" of USA and Canada. There are more than 400 references to his name in mathematical problem-solving situations in famous magazines from the whole world such as *American Mathematical Monthly*, *College Mathematics Journal*, *Mathematics Magazine*, *Crux Mathematicorum*. In 2009, he participated as an observer in the World Olympiad which was held in Germany, Bremen and in 2010 in the Balkan Olympiad Juniors in Romania. In 2013, he participated as second-in-command at the Balkan Olympiad in Cyprus and in 2014 also as second-in-command at the Balkan Olympiad Juniors held in Ohrid. His book *Beauty of proving* was very appreciated and became a best-seller in mathematical world.

DANIEL SITARU

Daniel Sitaru, born on 9 August 1963 in Craiova, Romania, is a teacher at National Economic College "Theodor Costescu" in Drobeta-Turnu Severin. He published 33 mathematical books, last two of these, *Math Phenomenon* and *Algebraic Phenomenon*, were very appreciated worldwide. He is the founding editor of *Romanian Mathematical Magazine*, an Interactive Mathematical Journal with 3.200.000 visitors in 2017 (www.ssmrmh.ro). Many problems from his books were published in famous journals such as *American Mathematical Monthly*, *Crux Mathematicorum*, *Math Problems Journal*, *The Pentagon Journal*, *La Gaceta de la RSME*, *SSMA Magazine*. He also published an impressive number of original problems in all mathematical journals from Romania (*GMB*, *Cardinal*, *Elipsa*, *Argument*, *Recreații Matematice*). His articles from *Crux Mathematicorum* and *The Pentagon Journal* were also very appreciated.

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Chapter I
PROBLEMS

Notations:

s — semiperimeter of ΔABC, F — area of ΔABC, R — circumradii, r — inradii,
h_a, h_b, h_c — altitudes, m_a, m_b, m_c — medians, s_a, s_b, s_c — symedians,
w_a, w_b, w_c — internal bisectors, r_a, r_b, r_c — exradii

1. Let a, b, c be positive real numbers. Prove that:

$$\frac{a^5}{a^2+ab+b^2}+\frac{b^5}{b^2+bc+c^2}+\frac{c^5}{c^2+ca+a^2}\geq abc.$$

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2. Prove that in any acute-angled ΔABC with length's sides $a \geq b \geq c$ the following relationship holds:

$$a^3+2b^3+c^3\leq (a+b)(ab+c^2)+(b-c)(bc+a^2).$$

Daniel Sitaru

3. Let a, b, c be positive real numbers with $a+b+c=3$. Prove that:

$$\frac{ab(b+1)}{c}+\frac{bc(c+1)}{a}+\frac{ca(a+1)}{b}\geq 6.$$

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4. In ΔABC; $M, N, P \in [BC]$. Prove that:

$$\sqrt[3]{AM\cdot AN\cdot AP}\left(\frac{1}{AM}+\frac{1}{AN}+\frac{1}{AP}\right)\leq \frac{5}{3}+\frac{2}{3}\left(\frac{b}{c}+\frac{c}{b}\right).$$

Daniel Sitaru

5. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}\geq \frac{3}{2}.$$

George Apostolopoulos

6. Prove that if $M \in (BC), N \in (AC), P \in (AB)$ then:

$$\frac{AM^2}{BN+CP}+\frac{BN^2}{CP+AM}+\frac{CP^2}{AM+BN}>\frac{F^2}{2s}\sum \frac{1}{bc\sin^2\frac{A}{2}}.$$

Daniel Sitaru

7. Let a, b, c be positive real numbers. Prove that:

$$\frac{(a^2-ab+b^2)^2}{(a+b)^4}+\frac{(b^2-bc+c^2)^2}{(b+c)^4}+\frac{(c^2-ca+a^2)^2}{(c+a)^4}\geq \frac{3}{16}.$$

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8. Let be $A', A'' \in (BC)$; $B', B'' \in (BC)$; $C', C'' \in (AB)$ in ΔABC; $AA' \cap BB' \cap CC' \neq \emptyset$. Prove that:

$$\frac{27[A'B'C']}{[A''B''C'']} \leq \left(\frac{BA'}{BA''} + \frac{CB'}{CB''} + \frac{AC'}{AC''}\right)^3.$$

Daniel Sitaru

9. Let a, b be positive real numbers with $a^2 + ab + b^2 = k^2, k > 0$. Prove that:

$$\sqrt{a+b} + \sqrt[4]{ab} \leq \frac{\sqrt{2}+1}{\sqrt[4]{3}} \cdot \sqrt{k}.$$

George Apostolopoulos

10. Prove that in any triangle ABC the following relationship holds:

$$3\left(a^a b^b c^c\right)^{\frac{1}{2s}} \geq \sqrt[9]{4RF} \sum \left(a^a b^b c^c\right)^{\frac{1}{3s}}.$$

Daniel Sitaru

11. Let a, b be positive real numbers such that $a^2 + ab + b^2 = 9$. Find the maximal value of expression:

$$(a+b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 16.$$

George Apostolopoulos

12. Find $x, y, z, t \in \left[0, \frac{\pi}{2}\right]$ such that:

$$\sqrt{\sin x \sin y \sin z \sin t} + \sqrt{\cos x \cos y \cos z \cos t} = 1.$$

Daniel Sitaru

13. Let x, y, z be positive real numbers. Prove that:

$$\sqrt[3]{(2x^3 + 3x^2 + 3x + 1)(2y^3 + 3y^2 + 3y + 1)(2z^3 + 3z^2 + 3z + 1)} \geq$$
$$\geq xyz + 8\sqrt{xyz}.$$

George Apostolopoulos

14. Prove that if $a \leq b \leq c$ in ΔABC then:

$$(m_b m_c)^{m_b - m_c}(m_a m_b)^{m_a - m_b} \geq (m_a m_c)^{m_a - m_c}.$$

Daniel Sitaru

15. Let x, y, z be positive real numbers with $xyz = 1$. Prove that:

$$\frac{\sqrt{x^4+1} + \sqrt{y^4+1} + \sqrt{z^4+1}}{x^2 + y^2 + z^2} \leq \sqrt{2}.$$

George Apostolopoulos

16. Prove that in any triangle ABC the following relationship holds:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq \sqrt[6]{4RF}\left(\sqrt[4]{ab} + \sqrt[4]{bc} + \sqrt[4]{ca}\right).$$

Daniel Sitaru

17. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\left(a^3(a+1) + b^3(b+1) + c^3(c+1)\right) \cdot \frac{(a^3+3)(b^3+3)(c^3+3)}{(a+1)(b+1)(c+1)} \geq 48.$$

George Apostolopoulos

18. Prove that if $a, b, c, d \in \mathbb{R}^*$ then:

$$(abc - ac - bc - ac)^2 \leq 4(1+a^2)(1+b^2)(1+c^2).$$

Daniel Sitaru

19. Let a, b, c be positive real numbers. Prove that:

$$\frac{3a^2+5ab+3b^2}{a^2+ab+b^2} + \frac{3b^2+5bc+3c^2}{b^2+bc+c^2} + \frac{3c^2+5ca+3a^2}{c^2+ca+a^2} \leq 11.$$

George Apostolopoulos

20. Prove that if $n \in \{1, 2\}$ then in any triangle ABC the following relationship holds:

$$3^{2-n}\left(\frac{a^{2n+1}}{m_a} + \frac{b^{2n+1}}{m_b} + \frac{c^{2n+1}}{m_c}\right) \geq 2\sqrt{3}(abc)^{\frac{2n}{3}}.$$

Daniel Sitaru

21. Let x, y be positive real numbers with $xy = 3$. Find the minimum value of expression: $\sqrt{x^2+1} + \sqrt{y^2+1}$.

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22. Prove that if $a, b, c \in (0, 1)$ are the length's sides in any triangle ABC then:

$$\frac{(s-2)^2 + r^2 + 4rR - 1}{(s-1)^2 + r^2 + 4R(r-s)} \geq \frac{3}{\sqrt[3]{(1-a)(1-b)(1-c)}}.$$

Daniel Sitaru

23. Let a, b, c be positive real numbers such that $a + b + c = 1$. Find the maximal value of the expression $A = \left(a - \frac{1}{2}\right)^3 + \left(b - \frac{1}{2}\right)^3 + \left(c - \frac{1}{2}\right)^3$.

George Apostolopoulos

24. Prove that in any triangle ABC the following relationship holds:

$$6\sum \frac{a}{2a^2+bc} \leq s\sum \frac{1}{m_a m_b}.$$

Daniel Sitaru

25. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that:

$$\frac{a^{2n+1}}{a^n + (n-1)b^n} + \frac{b^{2n+1}}{b^n + (n-1)c^n} + \frac{c^{2n+1}}{c^n + (n-1)a^n} + \frac{n-1}{n}(a^n b + b^n c + c^n a) \geq$$
$$\geq \frac{1}{3^n}$$

for all integers n with $n \geq 1$.

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26. Prove that if $a, b, c, d \in (0, \infty)$; $(a^2 + b^2)(c^2 + d^2) \neq 0$ then:
$$4((ac + bd)^6 + (ad - bc)^6) \geq (a^2 + b^2)^3(c^2 + d^2)^3.$$
Daniel Sitaru

27. Let a, b, c be positive real numbers such that $ab + bc + ca = 2$. Prove that:
$$\frac{a^4 + b^4}{(a + b)^2} + \frac{b^4 + c^4}{(b + c)^2} + \frac{c^4 + a^4}{(c + a)^2} \geq 1.$$
George Apostolopoulos

28. Prove that if $0 < x < y < z < \frac{\pi}{2}$ then:
$$(x + y)\sin z + (x - z)\sin y < (y + z)\sin x.$$
Daniel Sitaru

29. Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$.
Find the maximum value of expression:
$$A = (a + b + c)^{-2} + (b + c + 2)^{-2} + (c + a + 2)^{-2}.$$
George Apostolopoulos

30. Prove that in any triangle ABC the following relationship holds:
$$\sum \sin A\,|\cos A| \leq \sum (\sin B + \sin C)|\cos A - \sin A|.$$
Daniel Sitaru

31. Find all triples (x, y, z) of positive real numbers such that $xyz = 1$ and
$$\frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} = 3.$$
George Apostolopoulos

32. Prove that in any acute-angled triangle ABC the following relationship holds:
$$\cos\left(\frac{\pi}{4} - A\right) + \cos\left(\frac{\pi}{4} - B\right) + \cos\left(\frac{\pi}{4} - C\right) > \frac{2F}{R^2}.$$
Daniel Sitaru

33. Let a, b, c and d be real numbers, such that $a + b + c + d = +1$ or -1. Prove that:
$$\sqrt[4]{a^4 + b^4 + c^4} + \sqrt[4]{b^4 + c^4 + d^4} + \sqrt[4]{c^4 + d^4 + a^4} + \sqrt[4]{d^4 + a^4 + b^4} \geq \sqrt[4]{3}.$$
George Apostolopoulos

34. Prove that in any acute-angled triangle ABC the following relationship holds:
$$\sum AB\,(\sin^{2n+1} C + \sin^{2n+1} C) \geq \frac{9\sqrt{2}ABC}{\pi \cdot 2^n};\ n \in \mathbb{N}^*.$$
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35. Let a, b and c be positive real numbers, such that $\sqrt[4]{\frac{a}{8}}+\sqrt[4]{\frac{b}{8}}+\sqrt[4]{\frac{c}{8}}=1$.

Prove that:

$$\sqrt[4]{a+b}+\sqrt[4]{b+c}+\sqrt[4]{c+a}\geq 2.$$

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36. Prove that in any triangle ABC the following relationship holds:

$$\sum\frac{a^2(b^2+c^2-a^2)}{b^2c^2}\geq 64F^2(1-\cos^2A-\cos^2B-\cos^2C).$$

Daniel Sitaru

37. Let a, b, c and d be positive real numbers such that:

$$\sqrt[4]{\frac{a}{27}}+\sqrt[4]{\frac{b}{27}}+\sqrt[4]{\frac{c}{27}}+\sqrt[4]{\frac{d}{27}}=1.$$

Prove that:

$$\sqrt[4]{a+b+c}+\sqrt[4]{b+c+d}+\sqrt[4]{c+d+a}+\sqrt[4]{d+a+b}\geq 3.$$

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38. Let be $ABCD$ a trapezoid. If $AB \parallel CD$; $AB=a$; $CD=b$; $AD=c$; $BC=d$; $a>b$ then:

$$\text{Area }[ABCD]<\frac{(a+b)(a-b+c+d)^2}{16(a-b)}.$$

Daniel Sitaru

39. Let a, b, c be positive real numbers. Prove that:

$$\frac{a}{2a+b+c}+\frac{b}{2b+c+a}+\frac{c}{2c+a+b}\leq\frac{a}{2b+2c}+\frac{b}{2c+2a}+\frac{c}{2a+2b}.$$

George Apostolopoulos

40. Prove that in any acute-angled triangle ABC the following relationship holds:

$$\sqrt{(2^{h_a}+2^{h_b}+2^{h_c})(2^{m_a}+2^{m_b}+2^{m_c})}<2^a+3^b+4^c.$$

Daniel Sitaru

41. Let a, b, c be positive real numbers such that $a=b=c=1$.
Prove that:

$$(ab+bc+ca-3abc)\cdot\sqrt[4]{\frac{1}{a^3(b+c)^4}+\frac{1}{b^3(c+a)^4}+\frac{1}{c^3(a+b)^4}}\geq 1.$$

George Apostolopoulos

42. Prove that in any acute-angled triangle ABC the following relationship holds:

$$2s+\sqrt{\sum(a^2+2ab\cos(A-B))}\geq\sum\sqrt{a^2+2ab\cos(A-B)+b^2}.$$

Daniel Sitaru

43. Let a, b, c be positive real numbers, such that $ab + bc + ca = \frac{1}{3}$. Prove that:

$$\frac{a^4 + a^2b^2 + b^4}{a^2 + b^2} + \frac{b^4 + b^2c^2 + c^4}{b^2 + c^2} + \frac{c^4 + c^2a^2 + a^4}{c^2 + a^2} \geq \frac{1}{2}.$$

George Apostolopoulos

44. Prove that in any acute-angled triangle ABC the following relationship holds:

$$2\sum \left|\cos\frac{A - B}{2}\right| \leq 3 + \sqrt{3 + 2\sum \cos(A - B)}.$$

Daniel Sitaru

45. Let a, b and c be positive real numbers. Prove that:

$$\frac{(a^3 + b^3)^3}{(a + b)(a^2 + b^2)} + \frac{(b^3 + c^3)^3}{(b + c)(b^2 + c^2)} + \frac{(c^3 + a^3)^3}{(c + a)(c^2 + a^2)} \geq 6a^2b^2c^2.$$

George Apostolopoulos

46. Prove that in any non-isosceles triangle ABC the following relationship holds:

$$\frac{2\big((h_b - h_a)(m_c - m_a) - (h_c - h_a)(m_b - m_c)\big)^2}{h_a^2 + h_b^2 + h_c^2 - h_ah_b - h_bh_c - h_ch_a} \leq 3(a^2 + b^2 + c^2).$$

Daniel Sitaru

47. Let a, b, c be positive real numbers, such that $abc = 1$. Prove that:

$$\frac{(a^2 + b^2)^3}{a^3 + b^3} + \frac{(b^2 + c^2)^3}{b^3 + c^3} + \frac{(c^2 + a^2)^3}{c^3 + a^3} \geq 12.$$

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48. Let be $P \in Int\ (\Delta ABC)$; $PA = x$; $PB = y$; $PC = z$. Prove that:

$$(b^2y^2 + c^2z^2 - a^2x^2)(c^2z^2 + a^2x^2 - b^2y^2)(a^2x^2 + b^2y^2 - c^2z^2) \leq$$
$$\leq (abcxyz)^2.$$

Daniel Sitaru

49. Let a, b, c be positive real numbers. Prove that:

$$\frac{a^2 + ab}{\left(\sqrt{ca} + b\right)^2} + \frac{b^2 + bc}{\left(\sqrt{ab} + c\right)^2} + \frac{c^2 + ca}{\left(\sqrt{bc} + a\right)^2} \geq \frac{3}{2}.$$

George Apostolopoulos

50. Let be $P \in Int(\Delta ABC)$; $PA = x$; $PB = y$; $PC = z$. Prove that:

$$27(by + cz - ax)(ax + cz - by)(ax + by - cz) \leq (ax + by + cz)^3.$$

Daniel Sitaru

51. Prove that:

$$\frac{1}{(a + 1)^2} + \frac{1}{(b + 1)^2} + \frac{1}{(c + 1)^2} \geq \frac{2(a + b + c) + 6}{a + b + c - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}$$

for all real numbers a, b, c, each different from -1 and satisfying $abc = -1$. When the equality holds?

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52. Prove that if $a, b, c \in \left(0, \frac{\pi}{2}\right]$ then:

$$\frac{6}{\sqrt[3]{abc}} \leq \pi \sum \frac{\sin a}{b^2} \leq \pi \sqrt{3 \sum \frac{a^2}{b^4}}.$$

Daniel Sitaru

53. Let x, y and z be positive real numbers such that $x + y + z = \frac{3}{2}$. Find the minimum value of the expression $A = x^3 + y^3 + z^3 + x^2 + y^2 + z^2$.

George Apostolopoulos

54. Prove that if $a, b, c, x \in \mathbb{R}$ then:

$$a^2 + b^2 + c^2 + (\sin x + \cos x + \sin x \cos x)(ab + bc + ca) \geq 0.$$

Daniel Sitaru

55. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\frac{\left(2(a^4+1)(b^4+1)(c^4+1)\right)^{\frac{1}{2}}}{(a^2-a+1)(b^2-b+1)(c^2-c+1)} - \frac{1}{2}\left(\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}\right) \leq 1.$$

George Apostolopoulos

56. Let be $A_i(x_i, y_i);\ i \in \overline{1,3}$ the vertices of $\Delta A_1A_2A_3$ with a, b, c length's sides. Prove that:

$$\sum \left(\left(b(x_2 - x_1) + c(x_3 - x_1)\right)^2 + (b(y_2 - y_1) + c(y_3 - y_1)^2)\right) >$$

$$> 3 \left\| \begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix} \right\|^2.$$

Daniel Sitaru

57. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\left(a^3(a+1) + b^3(b+1) + c^3(c+1)\right) \cdot \frac{(a^3+3)(b^3+3)(c^3+3)}{(a+1)(b+1)(c+1)} \geq 48.$$

George Apostolopoulos

58. Prove that in any acute-angled triangle ABC the following relationship holds:

$$\left(\frac{a^2}{\tan B} + \frac{b^2}{\tan C} + \frac{c^2}{\tan A}\right)\left(\frac{a^2}{\tan C} + \frac{b^2}{\tan A} + \frac{c^2}{\tan B}\right) \geq a^2b^2 + b^2c^2 + c^2a^2.$$

Daniel Sitaru

59. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that:

$$\frac{1}{a^3 + (b+c)^3} + \frac{1}{b^3 + (c+a)^3} + \frac{1}{c^3 + (a+b)^3} \geq 9abc.$$

George Apostolopoulos

60. Prove that in any acute-angled triangle ABC the following relationship holds:

$$\sum a^2(3b^2+2bc+3c^2-3a^2)\geq 80F^2.$$

Daniel Sitaru

61. Let a,b,c be positive real numbers such that $abc=1$ and let x,y,z be real numbers such that $xy+yz+zx\geq 3$. Prove that:

$$(3x^2+10)\frac{a^3+b^3}{a^2+ab+b^2}+(3y^2+10)\frac{b^3+c^3}{b^2+bc+c^2}+(3z^2+10)\frac{c^3+a^3}{c^2+ca+a^2}\geq$$
$$\geq 26.$$

George Apostolopoulos

62. Prove that if $a,b,c\in\mathbb{R}$; $a+b+c=4$ then:

$$\sum(a\sin^2 a+b\sin^2(b+c))<5+\sum a\cos^2 b.$$

Daniel Sitaru

63. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$2\left(\sum_{k=1}^{n}\left(k+\frac{1}{a^k}\right)\left(k+\frac{1}{b^k}\right)\left(k+\frac{1}{c^k}\right)+1\right)^{\frac{1}{2}}\geq n^2+3n+2.$$

George Apostolopoulos

64. Prove that in any triangle ABC the following relationship holds:

$$\frac{1}{r^3}\sum a^3\cos B\cos C\geq 16\left(\sum\sin A\right)\left(\sum\cos^2 A\right).$$

Daniel Sitaru

65. Let a,b,c,d be positive real numbers such that $abcd=1$. Prove that:

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{d^3}+\frac{d^3}{a^3}+4\geq 2(a+b+c+d).$$

George Apostolopoulos

66. Prove that if $a,b,c,d\in\mathbb{R}$; $(a^2+b^2)(c^2+d^2)\neq 0$ then:

$$\left|\frac{a(c+d)-b(c-d)}{\sqrt{(a^2+b^2)(c^2+d^2)}}\right|\leq\left|1+\frac{(ad-bc)(ac+bd)}{(a^2+b^2)(c^2+d^2)}\right|.$$

Daniel Sitaru

67. Let a,b,c,d be positive real numbers with $abcd=16$. Prove that:

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{d^3}+\frac{d^3}{a^3}+4\geq a+b+c+d.$$

George Apostolopoulos

68. Prove that if $A,B,C,D>0$; $A+B+C+D=\frac{\pi}{4}$ and

$$\Omega_1=\sum\tan A+\sum\tan A\tan B-\sum\tan A\tan B\tan C$$

$$\Omega_2 = \frac{\sin^2(A+B)\sin^2(C+D)}{\cos^2 A\cos^2 B\cos^2 C\cos^2 D}$$

then: $16(\Omega_1 - 1) \le \Omega_2$.

Daniel Sitaru

69. Let x, y, z be positive real numbers with $x + y + z = 3$, Prove that:

$$24xyz \le 8\left(\sum yz\right) \le \sum \frac{(x+y)^4}{x^2+y^2} \le 8\left(\sum x^2\right) \le 8\left(\frac{3}{xyz}\right) \le 8\left(\sum \frac{1}{x^2}\right)$$

where the sums are over all cyclic permutations of (x, y, z).

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70. Let be $ABCD$ a convex quadrilater with $m\left(\widehat{BAD}\right) = m\left(\widehat{DCB}\right) = 90^\circ$ and $M \in (AB); Q \in (AD);\ N \in (BC);\ P \in (DC)$. Prove that:

$$AB \cdot AM + AD \cdot AQ + CN \cdot CB + CP \cdot CD < 2BD^2.$$

Daniel Sitaru

71. Let a, b, c be nonnegative real numbers such that $a + b + c = 4$. Prove that:

$$\frac{a^2b}{3a^2+b^2+4ac} + \frac{b^2c}{3b^2+c^2+4ab} + \frac{c^2a}{3c^2+a^2+4bc} \le \frac{1}{2}.$$

George Apostolopoulos

72. Let ABC be a triangle; $M \in (BC); N \in (AC);\ P \in (AB); \frac{MB}{MC} = \frac{NC}{NA} = \frac{PA}{PB}$.
Prove that if $AM = m; BN = n; CP = p$ then:

$$(m+n+p)^3 \ge 27(m+n-p)(m+p-n)(n+p-m).$$

Daniel Sitaru

73. Let a, b and c be positive real numbers such that $a + b + c = 3$. Prove that:

$$\sqrt{\frac{a}{b}+\frac{1}{a}} + \sqrt{\frac{b}{c}+\frac{1}{b}} + \sqrt{\frac{c}{a}+\frac{1}{c}} \ge 3\sqrt{2}.$$

George Apostolopoulos

74. In triangle $ABC; M \in (BC); N \in (AC); P \in (AB);\ \frac{MB}{MC} = \frac{NC}{NA} = \frac{PA}{PB}$.
Prove that:

$$AM^2(b^2+c^2-a^2) + BN^2(a^2+c^2-b^2) + CP^2(a^2+b^2-c^2) \ge$$
$$\ge 16F[ABC]F[MNP].$$

Daniel Sitaru

75. Let $a_i, i = 1, 2, \ldots, n$ be positive real numbers with $\prod_{i=1}^{n} a_i = 1$. Prove that:

a. $\displaystyle\sum_{i=1}^{n} \frac{3a_i^2 + a_{i+1}^2}{a_i + a_{i+1}} \ge 2n$ b. $\displaystyle\sum_{i=1}^{n} \frac{a_i^2}{2a_i + a_{i+1}} \ge \frac{n}{3}$

where $a_{n+1} = a_1$.

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76. Solve the following equation:

$$(\sin^2 x)^{\frac{1}{\sin^2 x}} \cdot (\cos^2 x)^{\frac{1}{\cos^2 x}} = \frac{1}{16}.$$

Daniel Sitaru

77. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that:

$$\sqrt[3]{\frac{a^3 + b^3}{2}} + \sqrt[3]{\frac{b^3 + c^3}{2}} + \sqrt[3]{\frac{c^3 + a^3}{2}} \geq 1.$$

George Apostolopoulos

78. Prove that if $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{1}{1 - \sin x \sin y} + \frac{1}{1 - \sin y \sin z} + \frac{1}{1 - \sin x \sin z} \leq \frac{1}{\cos^2 x} + \frac{1}{\cos^2 y} + \frac{1}{\cos^2 z}.$$

Daniel Sitaru

79. Let a, b, c be positive real numbers with $abc = 1$. Prove that:

$$\frac{(a+b)^7 - a^7 - b^7}{(a+b)^5 - a^5 - b^5} + \frac{(b+c)^7 - b^7 - c^7}{(b+c)^5 - b^5 - c^5} + \frac{(c+a)^7 - c^7 - a^7}{(c+a)^5 - c^5 - a^5} \geq \frac{63}{5}.$$

George Apostolopoulos

80. Prove that in any ΔABC the following relationship holds:

$$a^{2015}(2s - 3a) + b^{2015}(2s - 3b) + c^{2015}(2s - 3c) \leq 0.$$

Daniel Sitaru

81. Let a, b, c be positive real numbers with $a + b + c = 3$. Prove that:

$$\left(1 + \frac{3}{b}\right)^{a^3} \cdot \left(1 + \frac{3}{c}\right)^{b^3} \cdot \left(1 + \frac{3}{a}\right)^{c^3} \geq 64.$$

George Apostolopoulos

82. In ΔABC; $B \in (MC)$; $C \in (BN)$; $AM = AN$. In these conditions:

$$(MB - NC)(AB - AC) \leq 0.$$

Daniel Sitaru

83. Let $a_i, i = 1, 2, \ldots, n$ be positive real numbers such that $\sum_{i=1}^{n} a_i^2 \leq n$, where $a_{n+1} = a_1$. Prove that:

$$\prod_{i=0}^{n} \left(1 + \frac{1}{a_i \cdot a_{i+1}}\right)^{a_i^2} \geq \sqrt[n]{2^{\left(\sum_{i=1}^{n} a_i\right)^2}}.$$

George Apostolopoulos

84. Prove that in any ABC triangle the following relationship holds:

$$\frac{16}{(a+3)(b+5)(c+7)} \leq \frac{1}{4RF} + \frac{1}{105}.$$

Daniel Sitaru

85. Let a, b and c be positive real numbers. Prove that:

$$\frac{(2a+b)(2b+c)}{(a+2b)(b+2c)}+\frac{(2b+c)(2c+a)}{(b+2c)(c+2a)}+\frac{(2c+a)(2a+b)}{(c+2a)(a+2b)}<\frac{25}{3}.$$

George Apostolopoulos

86. Let H be the orthocenter of ABC acute-angled triangle. Prove that:

$$4a^2b^2c^2 \le \sum_{cyc}(acAH+bcBH)^2.$$

Daniel Sitaru

87. Let a, b and c be positive real numbers. Prove that:

a. $\sqrt{\frac{a+2b}{2a+3b+c}}+\sqrt{\frac{b+2c}{a+2b+3c}}+\sqrt{\frac{c+2a}{3a+b+2c}}\le\frac{3\sqrt{2}}{2}$;

b. $\left(\frac{2a+b}{a+2b}\right)^2+\left(\frac{2b+c}{b+2c}\right)^2+\left(\frac{2c+a}{c+2a}\right)^2>\frac{4}{3}$.

George Apostolopoulos

88. Prove that in any ABC acute-angled triangle we have:

$$27R^3\cos A\cos B\cos C\le(R+r)^3$$

Daniel Sitaru

89. Let a, b and c be positive real numbers. Prove that:

a. $\frac{a+2b}{2a+3b+c}+\frac{b+2c}{a+2b+3c}+\frac{c+2a}{3a+b+2c}\le\frac{3}{2}$;

b. $\frac{2a+b}{a+2b}+\frac{2b+c}{b+2c}+\frac{2c+a}{c+a}>2$.

George Apostolopoulos

90. Solve the following system:

$$\begin{cases} xyz=6 \\ \arctan x+\arctan y+\arctan z=\pi. \\ xy+xz+yz=11 \end{cases}$$

Daniel Sitaru

91. Let a, b and c be positive real numbers such that $a^2+b^2+c^2=3\sqrt{2}$. Prove that:

$$\frac{1}{a^4+b^4+4}+\frac{1}{b^4+c^4+4}+\frac{1}{c^4+a^4+4}\le\frac{3}{8}.$$

George Apostolopoulos

92. Let H, G be the orthocenter and the centroid of ABC acute-angled triangle. Prove that:

$$\sum_{cyc}\left(\frac{AH}{AG}\right)^2\ge\frac{108r^2}{a^2+b^2+c^2}.$$

Daniel Sitaru

93. Let a, b, c be positive real numbers such that $a+b+c=3$.

a. Find the maximum value of the expression
$$A=\frac{a^3+b^3+c^3}{(a^2+b^2+c^2)\cdot(a^4+b^4+c^4)}.$$
b. Find the minimum value of the expression
$$B=a^2(a^2-a+1)+b^2(b^2-b+1)+c^2(c^2-c+1).$$
George Apostolopoulos

94. Prove that if $x, y, z \in \left(0, \frac{\pi}{2}\right)$ and $x+y+z=1$ then:
$$2(\tan x+\tan y+\tan z)\geq\frac{1}{1-(xy+yz+zx)}.$$
Daniel Sitaru

95. Let a, b, c be positive real numbers with $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Prove that:
$$ab(a+b)+bc(b+c)+ca(c+a)\geq\frac{2}{3}(a^2+b^2+c^2)+4abc.$$
George Apostolopoulos

96. Prove that:
$$\frac{1}{\sin\frac{\pi}{2016}}+\frac{1}{2^2\sin\frac{\pi}{2015}}+\frac{1}{3^2\sin\frac{\pi}{2014}}+\cdots+\frac{1}{2015^2\sin\frac{\pi}{2}}>\frac{2015}{2016}.$$
Daniel Sitaru

97. Prove that:
$$(\sin^2\theta)^{\cos^2\theta}+\frac{\cos^2\theta}{(2-\cos^2\theta)^{\sin^2\theta}}\geq 1, \theta\in\mathbb{R}.$$
George Apostolopoulos

98. Prove that:
$$\frac{1}{2015}\sin\frac{1}{2}+\frac{1}{2014}\sin\frac{1}{3}+\cdots+\frac{1}{2}\sin\frac{1}{2015}<\frac{2014}{2015}$$
Daniel Sitaru

99. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that:
$$\sqrt{\frac{ab}{bc+a}}+\sqrt{\frac{bc}{ca+b}}+\sqrt{\frac{ca}{ab+c}}\leq\frac{1}{2(ab+bc+ca)}.$$
George Apostolopoulos

100. Prove that $(\forall)x\in\mathbb{R}$:
$$\sin x+\cos x+\sin x\cos x\leq\frac{1}{2}+\sqrt{2}.$$
Daniel Sitaru

101. Let $a_1, a_2, a_3, \dots, a_n$ be real numbers such that $a_1 > a_2 > a_3 > \cdots > a_n$. Prove that:

$$\frac{1}{a_1 - a_2} + \frac{1}{a_2 - a_3} + \cdots + \frac{1}{a_{n-1} - a_n} + a_1 - a_n \geq 2(n-1).$$

George Apostolopoulos

102. Prove that in any ABC acute-angled triangle the following relationship holds:

$$2\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) + \tan A \tan B \tan C \geq 9\sqrt{3}.$$

Daniel Sitaru

103. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that:

$$\left((ab)^{\frac{2}{3}} + (bc)^{\frac{2}{3}} + (ca)^{\frac{2}{3}}\right)^{\frac{1}{2}} < \frac{2 + \sqrt{3}}{3}.$$

George Apostolopoulos

104. Prove that:

$$\sqrt{1 - \cos\frac{2\pi}{35}} + \sqrt{1 - \cos\frac{2\pi}{63}} > \sqrt{1 - \cos\frac{4\pi}{45}}.$$

Daniel Sitaru

105. Let a, b, c be positive real numbers such that $abc = k^3$. Prove that:

$$\frac{1}{a^4 + 2k^4} + \frac{1}{b^4 + 2k^4} + \frac{1}{c^4 + 2k^4} \leq \frac{1}{k^4}.$$

George Apostolopoulos

106. Prove that in any ABC triangle the following relationship holds:

$$\sum\left(a^{2b} + b^{2a}\right) > \frac{3}{2}.$$

Daniel Sitaru

107. If the roots of the equation $ax^3 + bx^2 + cd + d = 0$; $a, b, c, d \in \mathbb{R}$ with $a > 0$, are all nonnegative. Prove that:

$$d \leq \frac{4abc - b^3}{9a^2}.$$

George Apostolopoulos

108. Prove that in any ABC triangle the following relationship holds:

$$\frac{\sum_{cyc}\sqrt{a^3 A}}{\sum_{cyc}\sqrt[3]{a^4 A^2}} \geq \frac{\sum_{cyc}\sqrt[3]{a^2 A^4}}{\sum_{cyc}\sqrt{a A^3}}.$$

Daniel Sitaru

109. Let x, y, z be positive real numbers such that $x + y + z = xyz$. Find

a. the minimum value of expression

$$A = \sqrt{\frac{1}{3}x^4 + 1} + \sqrt{\frac{1}{3}y^4 + 1} + \sqrt{\frac{1}{3}z^4 + 1};$$

b. the maximum value of expression $B = x^2y^2z^2 - 2(x^2 + y^2 + z^2)$.

George Apostolopoulos

110. Prove that in any ABC triangle the following relationship holds:

$$\sin^2 A + \sin^2 B + \sin C \le 2\sqrt{1 + \cos(A - B)\cos C}.$$

Daniel Sitaru

111. Let x, y, z be positive real numbers such that $x + y + z = xyz$. Find the maximum value of expression $A = 2(xy + yz + zx) - x^2 - y^2 - z^2$.

George Apostolopoulos

112. Prove that in any ABC triangle the following relationship holds:

$$\prod \frac{ab}{w_a(a + b)\cos\frac{c}{2}} \ge 1.$$

Daniel Sitaru

113. Let a, b be distinct real numbers such that:

$$a^4 + b^4 - 3(a^2 + b^2) + 8 \le 2(a + b)(2 - ab).$$

Find the value of the expression $A = (ab)^n + (ab + 1)^n + (ab + 2)^n$, where n is a positive integer.

George Apostolopoulos

114. Prove that $(\forall)x, y \in \mathbb{R}$:

$$\sin^2 x + \sin^2 y + \sin(x + y) \le 2\sqrt{1 - \cos(x + y)\cos(x - y)}.$$

Daniel Sitaru

115. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\frac{a^{n+2}}{a^n + (n - 1)b^n} + \frac{b^{n+2}}{b^n + (n - 1)c^n} + \frac{c^{n+2}}{c^n + (n - 1)a^n} \ge \frac{3}{n}$$

for each positive integer n.

George Apostolopoulos

116. Prove that:

$$\left(\frac{1}{3} + \frac{1}{6\sin\frac{\pi}{5}}\right)\left(\frac{1}{6} + \frac{1}{3\cos\frac{\pi}{5}}\right) > \frac{5}{18}.$$

Daniel Sitaru

117. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that:

$$\left(\frac{a^4 + a^2b^2 + b^4}{a^2 + ab + b^2}\right)^2 + \left(\frac{b^4 + b^2c^2 + c^4}{b^2 + bc + c^2}\right)^2 + \left(\frac{c^4 + c^2a^2 + a^4}{c^2 + ca + a^2}\right)^2 \ge 3.$$

George Apostolopoulos

118. Prove that in any acute-angled ABC triangle we have the following relationship:

$$\frac{\tan A}{A}+\frac{\tan B}{B}+\frac{\tan C}{C}>\frac{A}{\sin A}+\frac{B}{\sin B}+\frac{C}{\sin C}.$$

Daniel Sitaru

119. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^3+b^3}{a^2+b^2+abc}+\frac{b^3+c^3}{b^2+c^2+abc}+\frac{c^3+a^3}{c^2+a^2+abc}\geq 2.$$

George Apostolopoulos

120. Prove that if $0<x\leq y\leq z<\frac{\pi}{2}$ then:

$$\left|\tan^3\left(\sqrt[3]{xyz}\right)-\tan x\tan y\tan z\right|<\tan^3 z-\tan^3 x.$$

Daniel Sitaru

121. Let a, b, c be positive real numbers such that $ab+bc+ca=18abc$.
Prove that:

$$\frac{a^2b}{a^2+4ab+b^2}+\frac{b^2c}{b^2+4bc+c^2}+\frac{c^2a}{c^2+4ca+a^2}\leq a^2+b^2+c^2.$$

George Apostolopoulos

122. Find $x, y, z, t\in(0,\pi)$ such that:

$$\begin{cases}\sin x\cos t+\sin t\cos y+\sin y\cos z=\frac{1}{2}\\ \sin x\cos y+\sin y\cos z+\sin z\cos x=\frac{3}{2}\end{cases}.$$

Daniel Sitaru

123. Let a, b, c be positive real numbers such that $a^4+b^4+c^4=3$. Prove that:

$$ab\left(\frac{b^3+1}{b^2+1}\right)^4+bc\left(\frac{c^3+1}{c^2+1}\right)^4+ca\left(\frac{a^3+1}{a^2+1}\right)^4\geq 3abc.$$

George Apostolopoulos

124. Prove that in any acute-angled ABC triangle:

$$\frac{27m_a^2m_b^2m_c^2}{F^6}\leq\left(\frac{1}{(s-a)^2}+\frac{1}{(s-b)^2}+\frac{1}{(s-c)^2}\right)^3.$$

Daniel Sitaru

125. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that:

$$\left(\frac{a^3+1}{a^2+1}\right)^2+\left(\frac{b^3+1}{b^2+1}\right)^2+\left(\frac{c^3+1}{c^2+1}\right)^2\geq ab+bc+ca.$$

George Apostolopoulos

126. Prove that in any ABC triangle the following relationship holds:

$$\sum \sqrt[3]{\tan A \tan B}\left(\sqrt[3]{\tan A} + \sqrt[3]{\tan B}\right) \le 2 \tan A \tan B \tan C.$$

Daniel Sitaru

127. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\sqrt{a + \frac{1}{b}} + \sqrt{b + \frac{1}{c}} + \sqrt{c + \frac{1}{a}} - \frac{a^2 + b^2 + c^2}{3} < \frac{7}{2}.$$

George Apostolopoulos

128. Prove that in any triangle ABC the following relationship holds:

$$(2s + F)\left(\frac{ab}{a + b} + \frac{cF}{c + F}\right) < (a + c)(a + c + F).$$

Daniel Sitaru

129. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$\left(\frac{a + 1}{a^3 + 1}\right)^2 + \left(\frac{b + 1}{b^3 + 1}\right)^2 + \left(\frac{c + 1}{c^3 + 1}\right)^2 \le a^4 + b^4 + c^4.$$

George Apostolopoulos

130. Prove that in any ABC triangle the following relationship holds:

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) > e^{\frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a}}.$$

Daniel Sitaru

131. Let x, y, z be positive real numbers such that $x + y + z = 3$. Prove that:

$$\left(\frac{x^4 + x^2 + 1}{x^2 + x + 1}\right)^2 + \left(\frac{y^4 + y^2 + 1}{y^2 + y + 1}\right)^2 + \left(\frac{z^4 + z^2 + 1}{z^2 + z + 1}\right)^2 \ge 3xyz.$$

George Apostolopoulos

132. Let be $x, y, z, t \in \mathbb{R}^*$ such that $\arctan x + \arctan y + \arctan z + \arctan t = \pi$. Prove that:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \ge \frac{4}{\sqrt[4]{(xyzt)^3}}.$$

Daniel Sitaru

133. Let x, y, z be positive real numbers such that $x + y + z = 3$. Prove that:

$$\sqrt[4]{xy^3} + \sqrt[4]{yz^3} + \sqrt[4]{zx^3} \le \sqrt{\frac{3}{x^2} + \frac{3}{y^2} + \frac{3}{z^2}}.$$

George Apostolopoulos

134. Prove that in any ABC triangle the following relationship holds:

$$a^4c + b^4a + c^4b \ge 24\sqrt[3]{2}(RF)^{\frac{5}{3}}.$$

Daniel Sitaru

135. Find the minimum value of expression

$$A = \frac{1}{x+y+z}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right),$$

when $x, y, z > 0$ and $x^2+y^2+z^2=1$. (x, y, z are real numbers).

George Apostolopoulos

136. Prove that in any ABC triangle the following relationship holds:

$$\sum (a+b)(a+c) \geq 8\sqrt{RF}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).$$

Daniel Sitaru

137. Find all possible pairs (x, y) of integers satisfying

$$x^4 + (4-3x^2)(y^2-y-1) = 2.$$

George Apostolopoulos

138. Prove that in any ABC triangle the following relationship holds:

$$\frac{aA^5+bB^5+cC^5}{aA^4+bB^4+cC^4} \leq \frac{aA^7+bB^7+cC^7}{aA^6+bB^6+cC^6}.$$

Daniel Sitaru

139. Find all pairs of positive integers x, y satisfying the equation

$$4x^2+3y^2-7xy-6x+5y=0.$$

George Apostolopoulos

140. Prove that in any ABC triangle the following relationship holds:

$$\sum (1+\sin A)^{2015} \geq \frac{4030s}{R} + 2^{2015}\sum \sin^{4030}\left(\frac{A}{2}\right).$$

Daniel Sitaru

141. Find all possible pairs (x, y) of integers satisfying

$$x^4-3x^2y^2+3xy^2-2x^3-9x^2+19y^2+10x+23=0.$$

George Apostolopoulos

142. Prove that in any triangle ABC the following relationship holds:

$$\left(\sum a\sin A\right)^2 + \left(\sum a\cos A\right)^2 \leq 4s^2.$$

Daniel Sitaru

143. Find all pairs of integers (x, y) such that

$$x^8+(y^2+y-1)(4-3x^4)=2.$$

George Apostolopoulos

144. Prove that in any ABC triangle the following relationship holds:

$$\frac{m_a m_b}{m_a+m_b}+\frac{m_b m_c}{m_b+m_c}+\frac{m_c m_a}{m_c+m_a} \leq p.$$

Daniel Sitaru

145. Find all possible pairs (x, y) of integers satisfying

$$x^8+(3y^2+3y-4)(1-x^4)=1.$$

George Apostolopoulos

146. Prove that if $A, B, C \in \left[1, \frac{\pi}{2}\right)$ then:

$$\prod\left(\frac{1+A}{A+\tan A}\right) \geq \frac{2}{ABC+\tan A\tan B\tan C}.$$

Daniel Sitaru

147. The equation $3x^3-3x+1=0$, has roots the numbers a, b, c.
Let $A=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $B=\frac{b}{a}+\frac{c}{a}+\frac{a}{c}$. Find $A \cdot B$.

George Apostolopoulos

148. Prove that in any ABC triangle the following relationship holds:

$$3(aA+bB+cC)<2\left(m_a^2+m_b^2+m_c^2\right)+6(A^2+B^2+C^2).$$

Daniel Sitaru

149. Let x, y, z be positive real numbers such that $x+y+z=\frac{1}{k}$. Prove that:

$$\sqrt{\frac{xy}{kxy+z}}+\sqrt{\frac{yz}{kyz+x}}+\sqrt{\frac{zx}{kzx+y}} \leq \frac{3}{2}\sqrt{x+y+z}.$$

George Apostolopoulos

150. Prove that in any ABC triangle the following relationship holds:

$$a^4+b^4+c^4 \geq 16F^2.$$

Daniel Sitaru

151. Let $x_i, y_i, z_i, i=1,2,3,4$ be positive real numbers. Prove that:

$$3 \cdot \prod_{i=1}^{4} \frac{x_i^4+y_i^4+z_i^4+1}{x_i+y_i+z_i} \geq \left(\frac{2}{3}\right)^8.$$

George Apostolopoulos

152. Prove that if ΔABC is an acute-angled triangle then:

$$\sum \frac{(\tan A)^{2n+1}}{\sqrt{\tan B\tan C}} \geq 3\left(\sum \tan A\right)^{\frac{2n}{3}}, n \in \mathbb{N}^*.$$

Daniel Sitaru

153. Let a, b be real numbers with $a \neq b, ab \geq -1$ such that

$$a^4+b^4-3(a^2+b^2)+8 \leq 2(a+b)(2-ab).$$

Prove that:

$$\frac{3}{2}\left(\sqrt{ab+1}+\sqrt{ab+2}+\cdots+\sqrt{ab+n}+\sqrt{ab+n+1}\right) > n^{\frac{3}{2}}, n \in \mathbb{N}^*.$$

George Apostolopoulos

154. Prove that in any ABC triangle the following relationship holds:

$$\pi^2\left(\sum\frac{a^3}{B^2}\right)\geq 27abc.$$

Daniel Sitaru

155. Let a,b,c be real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^2-a+1}{a^4+1}+\frac{b^2-b+1}{b^4+1}+\frac{c^2-c+1}{c^4+1}\geq\frac{9}{2(a^2+b^2+c^2)}.$$

George Apostolopoulos

156. Prove that in any ABC triangle the following relationship holds:

$$64\sum\frac{m_a^6}{b^4}\geq 27\sum a^2.$$

Daniel Sitaru

157. Let a,b,c be positive real numbers. Prove that:

$$\frac{ab-bc+ca}{a^2}+\frac{ab+bc-ca}{b^2}+\frac{-ab+bc+ca}{c^2}\leq 3.$$

George Apostolopoulos

158. Prove that in any ABC triangle the following relationship holds:

$$16\sum m_a^2m_b^2\geq 9abc(a+b+c).$$

Daniel Sitaru

159. Let a,b,c be positive real numbers. Prove that:

$$a^3+b^3+c^3+3\geq 3\big((a^2b+1)\cdot(b^2c+1)(c^2a+1)\big)^{\frac{1}{3}}.$$

George Apostolopoulos

160. Prove that in any ABC triangle the following relationship holds:

$$m_a^2h_a+m_b^2h_b+m_c^3h_c\geq\frac{9F}{2}\sqrt[3]{abc}.$$

Daniel Sitaru

161. Let a,b,c be positive real numbers. Prove that:

$$\frac{a^{2n+1}}{a^2+ab+b^2}+\frac{b^{2n+1}}{b^2+bc+c^2}+\frac{c^{2n+1}}{c^2+ca+a^2}\geq(abc)^{\frac{2n-1}{3}}$$

for each positive integer n.

George Apostolopoulos

162. Prove that in any ABC triangle the following relationship holds:

$$r\left(r_a\sqrt{w_am_a}+r_b\sqrt{w_bm_b}+r_c\sqrt{w_cm_c}\right)\leq sF.$$

Daniel Sitaru

163. Let a,b,c be positive real numbers with $a+b+c=1$. Prove that:

$$\left(1+\frac{1}{2a+b}\right)^c\cdot\left(1+\frac{1}{2b+c}\right)^a\cdot\left(1+\frac{1}{2c+a}\right)^b\geq 2.$$

George Apostolopoulos

164. Prove that in any ABC triangle the following relationship holds:

$$\sum \sqrt{w_a s_a r_a} \leq s\sqrt{s\sqrt{3}}$$

where s_a – is the length of simedian corresponding to the A vertex.

Daniel Sitaru

165. For any triangle with sides of lengths a, b and c, prove that:

$$\left(\frac{a}{b+c-a}\right)^n + \left(\frac{b}{c+a-b}\right)^n + \left(\frac{c}{a+b-c}\right)^n \geq 3, n \in \mathbb{N}.$$

George Apostolopoulos

166. Prove that in ΔABC with $a^2 + b^2 \neq c^2$ we have:

$$\frac{1}{(a^2+b^2-c^2)^2} + \frac{a^2+b^2}{4a^2b^2c^2} \geq \frac{81}{16(m_a^2+m_b^2+m_c^2)}.$$

Daniel Sitaru

167. Let a, b, c be positive real numbers satisfying $a + b + c = 3$. Prove that:

$$\sum_{cyclic} \left(\frac{2ab}{\sqrt{2a^2+b^2+c^2}} + \frac{ab}{b+c}\right) \leq \frac{9}{2}.$$

George Apostolopoulos

168. Prove that in any ABC triangle the following relationship holds:

$$\sum \frac{a(b+c)}{bc\cos^2\frac{A}{2}} \geq 8.$$

Daniel Sitaru

169. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$.
Prove that:

$$(3a^2+2)\frac{a^3+b^3}{a^2+ab+b^2} + (3b^2+2)\frac{b^3+c^3}{b^2+bc+c^2} + (3c^2+2)\frac{c^3+a^3}{c^2+ca+a^2} \geq$$
$$\geq 10abc.$$

George Apostolopoulos

170. Prove that in any ABC triangle the following relationship holds:

$$\left(\frac{1}{R^2} + \frac{2}{Rr} + \frac{1}{r^2}\right)(5r^2 + m_a^2 + m_b^2 + m_c^2) > 64.$$

Daniel Sitaru

171. Let a, b, c be positive real numbers such that $a + b + c = 1$, and $n \in \mathbb{N}$.
Prove that:

$$a^{2n+1} \cdot \frac{a+b}{\left(\sqrt{ca}+b\right)^2} + b^{2n+1} \cdot \frac{b+c}{\left(\sqrt{ab}+c\right)^2} + c^{2n+1} \cdot \frac{c+a}{\left(\sqrt{bc}+a\right)^2} \geq \frac{3^{1-2n}}{2}.$$

George Apostolopoulos

172. Prove that in any ABC triangle the following relationship holds:

$$8\sum(s-a)^3+2\geq 3\sqrt[3]{3abc}.$$

Daniel Sitaru

173. Let a,b,c be nonnegative real numbers, of which either two are not simultaneously zero. Prove that:

$$\frac{a+b}{\left(\sqrt{ca}+b\right)^2}+\frac{b+c}{\left(\sqrt{ab}+c\right)^2}+\frac{c+a}{\left(\sqrt{bc}+a\right)^2}\geq\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}.$$

George Apostolopoulos

174. Prove that in any ΔABC the following relationship holds:

$$ab+2bc<a^2+c^2+2ab\cos C+2bc\cos A.$$

Daniel Sitaru

175. Let a,b,c be positive real numbers. Prove that:

$$\left(\sum_{cyc}\frac{1}{\sqrt{\frac{a}{b}+\frac{b^2}{c^2}+\frac{c^3}{a^3}}}\right)^3\leq\frac{\sqrt{6}}{2}\left(\sum_{cyc}\sqrt{\frac{a^3+b^3}{c^3}}\right).$$

George Apostolopoulos

176. Prove that in any ABC triangle the following relationship holds:

$$\sqrt[6]{abc}\sum\left(\sin\frac{A}{4}+\cos\frac{A}{4}\right)\leq 3\sqrt{s}.$$

Daniel Sitaru

177. Let x,y and z be nonnegative real numbers such that $x+y+z=1$. Prove that:

$$xyz\leq\left(\frac{x^2+xy+y^2}{3}\right)^2+\left(\frac{y^2+yz+z^2}{3}\right)^2+\left(\frac{z^2+zx+x^2}{3}\right)^2\leq x^4+y^4+z^4.$$

George Apostolopoulos

178. Prove that in any ABC triangle the following relationship holds:

$$\sum\sqrt{\sin A\sin B}\leq\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}.$$

Daniel Sitaru

179. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{ab+bc+ca}{2}\geq\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}.$$

George Apostolopoulos

180. Prove that in any ΔABC the following relationship holds:

$$\sum\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}}\leq\cos\frac{\pi-A}{4}+\cos\frac{\pi-B}{4}+\cos\frac{\pi-C}{4}.$$

Daniel Sitaru

181. Let r_a, r_b, r_c be the exradii of a triangle ABC with inradius r and circumradius R. Prove that:

$$\frac{8}{9R^3} \le \frac{1}{r_a^3} + \frac{1}{r_b^3} + \frac{1}{r_c^3} \le \frac{81R^3 - 576r^3}{648r^6}.$$

George Apostolopoulos

182. Let $ABCD$ be a tetrahedron and $M \in (AB); N \in (BC); P \in (CD);\ Q \in (AD)$ such that:

$$\frac{AM}{MB} = x;\ \frac{BN}{NC} = \frac{1}{x+2};\ \frac{CP}{PD} = \frac{y+1}{3};\ \frac{DQ}{QA} = \frac{6}{y+2},$$

$x, y \in (0, \infty)$. Prove that if M, N, P, Q are coplanar then $x(y+1) \ge 4\sqrt{xy}$.

Daniel Sitaru

183. Let $a_i, i = 1, 2, \dots, n$ be positive real numbers such that $\sum_{i=1}^{n} a_i = n$. Prove that:

$$\sum_{i=1}^{n} \left(\frac{a_i^3 + 1}{a_i^2 + 1}\right)^4 \ge n.$$

George Apostolopoulos

184. Prove that if in ΔABC: $b\cos B + c\cos C = 2a\sin B\sin C$ then:

$$\frac{b^6}{c^4} > \frac{8(a-c)^3}{a}.$$

Daniel Sitaru

185. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that:

$$\sum_{cyc} a(3-a)\left(\frac{a^3+1}{a^2+1}\right)^4 \ge 6\prod_{cyc} a.$$

George Apostolopoulos

186. Prove that in any ΔABC the following relationship holds:

$$(5a^2 + 7b^2 + 3c^2)(7a^2 + 3b^2 + 5c^2) \ge 71(a^2b^2 + b^2c^2 + c^2a^2).$$

Daniel Sitaru

187. Let a, b, c be positive real numbers. Prove that:

a. $\dfrac{(a+b)^2}{a^2+ab+b^2} + \dfrac{(b+c)^2}{b^2+bc+c^2} + \dfrac{(c+a)^2}{c^2+ca+a^2} \le 4;$

b. $a + b + c = 3$, then $\dfrac{ab}{a+ab+b} + \dfrac{bc}{b+bc+c} + \dfrac{ca}{c+ca+a} \le 1.$

George Apostolopoulos

188. Prove that in any triangle ABC the following relationship holds:

$$\sin^2\left(\frac{\pi - A}{4}\right) + \sin^2\left(\frac{\pi - B}{4}\right) + \sin^2\left(\frac{\pi - C}{4}\right) > 1 - \frac{(\pi - A)(\pi - B)(\pi - C)}{32}.$$

Daniel Sitaru

189. Real numbers a, b, c satisfy $\sqrt{a^2+5a+4}+2\sqrt{b^2-5a+4}=\sqrt{5c^2+40}$. Prove that $a^2+b^2\geq c^2$.

George Apostolopoulos

190. Prove that in any ΔABC the following relationship holds:

$$6\sum\frac{a}{2a^2+bc}\leq s\sum\frac{1}{m_am_b}.$$

Daniel Sitaru

191. Let x, y, z be positive real numbers. Find the maximal value of expression:

$$A=\frac{x+2y}{2x+3y+z}+\frac{y+2z}{2y+3z+x}+\frac{z+2x}{2z+3x+y}.$$

George Apostolopoulos

192. Prove that in any ΔABC the following relationship holds:

$$\left|\sum\tan\frac{A-B}{2}\tan\frac{C}{2}\right|<1.$$

Daniel Sitaru

193. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that:

$$\frac{a^{2n+1}}{b+c}+\frac{b^{2n+1}}{c+a}+\frac{c^{2n+1}}{a+b}\geq\frac{9^{n+1}}{6}\cdot(abc)^{n+\frac{n}{3}}$$

for all $n\in\mathbb{N}$.

George Apostolopoulos

194. Prove that in any ΔABC the following relationship holds:

$$\left|\sum\frac{\sin A-\sin B}{\sin A+\sin B}\right|<1.$$

Daniel Sitaru

195. Let a, b, c be positive real numbers with $a+b+c=1$. Prove that:

$$a^4+b^4+c^4\geq abc.$$

George Apostolopoulos

196. Let $ABCD$ be a tetrahedron where:

$$AC=\sqrt{11};\ CD=3; AD=\sqrt{14}; AB=\sqrt{3};\ BC=2; BD=\sqrt{13}$$

Prove that $m\big(\sphericalangle(AB,CD)\big)>90°$.

Daniel Sitaru

197. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Find the maximum value of expression

$$A=\frac{(a+b)^5\cdot(a^3+b^3+c^3+5abc)}{(a^5+b^5)\cdot(a^4+b^4+c^4-2abc)}.$$

George Apostolopoulos

198. Prove that if $a, b, c\in\mathbb{R}$ then:

$$(2-a-b-c+abc)^2\leq(a^2+2)(b^2+2)(c^2+2).$$

Daniel Sitaru

199. Let ABC be a triangle with circumradius R and inradius r. Prove that:

$$27\left(\frac{r}{R}\right)^4 \leq \sin^4 A + \sin^4 B + \sin^4 C \leq \frac{27}{8}\left(1-\frac{r}{R}\right).$$

George Apostolopoulos

200. Prove that in any acute-angled ABC triangle, the following relationship

$$\sum(\sin 2A + \sin 2B)\left(\frac{1}{\sin 2A}+\frac{1}{\sin 2B}\right) \leq \sum(\tan A + \tan B)(\cot A + \cot B).$$

Daniel Sitaru

201. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that:

$$(a+b)(b+c)(c+a) - 2abc \leq 6.$$

George Apostolopoulos

202. Prove that if $x, y, z \in [0,\infty)$ then in ΔABC the following relationship holds:

$$\frac{R}{\sqrt{2F}}\big(x(b+c)+y(a+c)+z(a+b)\big) \geq \frac{ax}{\sqrt{\sin A}}+\frac{by}{\sqrt{\sin B}}+\frac{cz}{\sqrt{\sin C}}.$$

Daniel Sitaru

203. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that:

$$(a+b)(b+c)(c+a)+4abc \leq 12.$$

George Apostolopoulos

204. Prove that if $x, y \in \mathbb{R}; z \in [0,\infty)$ then:

$$z\sin(x-y)\cos(x+y)+\cos x\cos y \leq \cos(x+z)+\cos(y+z)+2\sqrt{2}z.$$

Daniel Sitaru

205. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$.
Prove that:

$$(a^3+b^3+c^3+d^3)\left(\frac{1}{a(bc+d)}+\frac{1}{b(cd+a)}+\frac{1}{c(da+b)}+\frac{1}{d(ab+c)}\right) \geq 8.$$

George Apostolopoulos

206. Prove that if a, b, c are the sides of a triangle then:

$$\sin^2 a + \sin^2 b + \sin^2 c \leq 4\sin s\sin(s-a)\sin(s-b)\sin(s-c).$$

Daniel Sitaru

207. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that:

$$3\left(\frac{a}{a^3+b^2+c}+\frac{b}{b^3+c^2+a}+\frac{c}{c^3+a^2+a}\right)^3+1 \leq \frac{4}{a^2b^2c^2}.$$

George Apostolopoulos

208. Prove that in ABC triangle we have:

$$8\sum\tan\frac{A-B}{2}\tan\frac{C}{2} \leq \left|1-\frac{b}{a}\right|\cdot\left|1-\frac{c}{b}\right|\cdot\left|1-\frac{a}{c}\right|.$$

Daniel Sitaru